

# Order Determination of Large Dimensional Dynamic Factor Model

By Z. D. Bai, Chen Wang, K. Krishnan Nair and Matthew Harding

Northeast Normal University, University of Cambridge,

Stanford University and Duke University

## Abstract

Consider the following dynamic factor model:  $\mathbf{R}_t = \sum_{i=0}^q \mathbf{\Lambda}_i \mathbf{f}_{t-i} + \mathbf{e}_t, t = 1, \dots, T$ , where  $\mathbf{\Lambda}_i$  is an  $n \times k$  loading matrix of full rank,  $\{\mathbf{f}_t\}$  are i.i.d.  $k \times 1$ -factors, and  $\mathbf{e}_t$  are independent  $n \times 1$  white noises. Now, assuming that  $n/T \rightarrow c > 0$ , we want to estimate the orders  $k$  and  $q$  respectively. Define a random matrix

$$\Phi_n(\tau) = \frac{1}{2T} \sum_{j=1}^T (\mathbf{R}_j \mathbf{R}_{j+\tau}^* + \mathbf{R}_{j+\tau} \mathbf{R}_j^*),$$

where  $\tau \geq 0$  is an integer. When there are no factors, the matrix  $\Phi_n(\tau)$  reduces to

$$\mathbf{M}_n(\tau) = \frac{1}{2T} \sum_{j=1}^T (\mathbf{e}_j \mathbf{e}_{j+\tau}^* + \mathbf{e}_{j+\tau} \mathbf{e}_j^*).$$

When  $\tau = 0$ ,  $\mathbf{M}_n(\tau)$  reduces to the usual sample covariance matrix whose ESD tends to the well known MP law and  $\Phi_n(0)$  reduces to the standard spike model. Hence the number  $k(q+1)$  can be estimated by the number of spiked eigenvalues of  $\Phi_n(0)$ . To obtain separate estimates of  $k$  and  $q$ , we have employed the spectral analysis of  $\mathbf{M}_n(\tau)$  and established the spiked model analysis for  $\Phi_n(\tau)$ .

## 1 Introduction

For a  $p \times p$  random Hermitian matrix  $\mathbf{A}$  with eigenvalues  $\lambda_j, j = 1, 2, \dots, p$ , the empirical spectral distribution (ESD) of  $\mathbf{A}$  is defined as

$$F^{\mathbf{A}}(x) = \frac{1}{p} \sum_{j=1}^p I(\lambda_j \leq x).$$

The limiting distribution  $F$  of  $\{F^{A_n}\}$  for a given sequence of random matrices  $\{A_n\}$  is called the limiting spectral distribution (LSD). Let  $\{\varepsilon_{it}\}$  be independent identically distributed (i.i.d) random variables with common mean 0, variance 1. Consider a high dimensional dynamic  $k$ -factor model with lag  $q$ , that is,  $\mathbf{R}_t = \sum_{i=0}^q \Lambda_i \mathbf{f}_{t-i} + \mathbf{e}_t, t = 1, \dots, T$ , where  $\Lambda_i$  is an  $n \times k$  loading matrix of full rank,  $\{\mathbf{f}_t\}$  are i.i.d.  $k \times 1$ -factors with common mean 0, variance 1, whereas  $\mathbf{e}_t$  corresponds to the noise component with  $\mathbf{e}_t = (\varepsilon_{1t}, \dots, \varepsilon_{nt})'$ . In addition, both components of  $\mathbf{e}_t$  and  $\mathbf{f}_t$  are assumed to have finite 4th moment.

This model can also be thought as an information-plus-noise type model (Dozier & Silverstein, 2007a, b; Bai & Silverstein, 2012). Here both  $n$  and  $T$  tend to  $\infty$ , with  $n/T \rightarrow c$  for some  $c > 0$ . Compared with  $n$  and  $T$ , the number of factors  $k$  and that of lags  $q$  are fixed but unknown. An interesting and important problem to economists is how to estimate  $k$  and  $q$ . To this end, define  $\Phi_n(\tau) = \frac{1}{2T} \sum_{j=1}^T (\mathbf{R}_j \mathbf{R}_{j+\tau}^* + \mathbf{R}_{j+\tau} \mathbf{R}_j^*)$ ,  $\gamma_t = \frac{1}{\sqrt{2T}} \mathbf{e}_t$  and  $\mathbf{M}_n(\tau) = \sum_{k=1}^T (\gamma_k \gamma_{k+\tau}^* + \gamma_{k+\tau} \gamma_k^*)$ ,  $\tau = 0, 1, \dots$ . Here  $*$  stands for the transpose and complex conjugate of a complex number and  $\tau$  is referred to be the number of lags. Denote

$$\begin{aligned} \Lambda &= (\Lambda_0, \Lambda_1, \dots, \Lambda_q)_{n \times k(q+1)}, \\ \mathbf{F}^\tau &= \begin{pmatrix} \mathbf{f}_{T+\tau} & \mathbf{f}_{T+\tau-1} & \cdots & \mathbf{f}_{\tau+1} \\ \mathbf{f}_{T+\tau-1} & \mathbf{f}_{T+\tau-2} & \cdots & \mathbf{f}_\tau \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{f}_{T+\tau-q} & \mathbf{f}_{T+\tau-1-q} & \cdots & \mathbf{f}_{\tau+1-q} \end{pmatrix}_{k(q+1) \times T}, \\ \mathbf{e}^\tau &= (\mathbf{e}_{T+\tau}, \mathbf{e}_{T+\tau-1}, \dots, \mathbf{e}_{\tau+1})_{n \times T}. \end{aligned}$$

Then we have that  $\Phi_n(\tau) = \frac{1}{2T} [(\Lambda \mathbf{F}^\tau + \mathbf{e}^\tau)(\Lambda \mathbf{F}^0 + \mathbf{e}^0)^* + (\Lambda \mathbf{F}^0 + \mathbf{e}^0)(\Lambda \mathbf{F}^\tau + \mathbf{e}^\tau)^*]$  and  $\mathbf{M}_n(\tau) = \frac{1}{2T} (\mathbf{e}^\tau \mathbf{e}^{0*} + \mathbf{e}^0 \mathbf{e}^{\tau*})$ .

Note that essentially,  $\mathbf{M}_n(\tau)$  and  $\Phi_n(\tau)$  are symmetrized auto-cross covariance matrices at lag  $\tau$  and generalize the standard sample covariance matrices  $\mathbf{M}_n(0)$  and  $\Phi_n(0)$ , respectively. The matrix  $\mathbf{M}_n(0)$  has been intensively studied in the literature and it is well known that the LSD has an MP law (Marčenko and Pastur, 1967). Readers may refer to Jin et al. (2014) and

Wang et al. (2015) for more details about the model.

To estimate  $k$  and  $q$ , the following method can be employed. First, note that when  $\tau = 0$  and  $\text{Cov}(\mathbf{f}_t) = \Sigma_{\mathbf{f}}$ , the population covariance matrix of  $\mathbf{R}_t$  is a *spiked population model* (Johnstone (2001), Baik and Silverstein (2006), Bai and Yao (2008)) with  $k(q+1)$  spikes. Therefore,  $k(q+1)$  can be estimated by counting the number of eigenvalues of  $\Phi_n(0)$  that are larger than some phase transition point. Next, the separated estimation of  $k$  and  $q$  can be achieved by investigating the spectral property of  $\mathbf{M}_n(\tau)$  for general  $\tau \geq 1$ , using the fact that the number of eigenvalues of  $\Phi_n(\tau)$  that lie outside the support of the LSD of  $\mathbf{M}_n(\tau)$  at lags  $1 \leq \tau \leq q$  is different from that at lags  $\tau > q$ . Thus, the estimates of  $k$  and  $q$  can be separated by counting the number of eigenvalues of  $\Phi_n(\tau)$  that lie outside the support of the LSD of  $\mathbf{M}_n(\tau)$  from  $\tau = 0, 1, 2, \dots, q, q+1, \dots$ .

Note that for the above method to work, the LSD of  $\mathbf{M}_n(\tau)$  for general  $\tau \geq 1$  must be known. This is derived in Jin et al. (2014). Moreover, it is required that no eigenvalues outside the support of the LSD of  $\mathbf{M}_n(\tau)$  so that if an eigenvalue of  $\Phi_n(\tau)$  goes out of the support of the LSD of  $\mathbf{M}_n(\tau)$ , it must come from the signal part. Wang et al. (2015) proved such phenomenon theoretically. Both results are included in Section 2 for readers' reference.

The rest of the paper is structured as follows: Some known results are given in Section 2. Section 3 presents truncation of variables and Section 4 estimates  $k(q+1)$ . The estimation of  $q$  is provided in Section 5, from the which the estimation of  $k$  can also be obtained. Section 6 discusses the case when the variance of the noise part is unknown. A simulation study is shown in Section 7 and some proofs are presented in Appendix.

Regarding the norm used in this paper, the norm applied to a vector is the usual Euclidean norm, with notation  $\|\cdot\|$ . For a matrix, two kinds of norm have been used. The operator norm, denoted by  $\|\cdot\|_o$ , is the largest singular value. For matrices of fixed dimension, the Kolmogorov norm, defined as the largest absolute value of all the entries, has been used, with notation  $\|\cdot\|_K$ .

## 2 Some known results

In this section, we present some known results.

**Lemma 2.1** (*Burkholder (1973)*). *Let  $\{X_k\}$  be a complex martingale difference sequence with respect to the increasing  $\sigma$ -fields  $\{\mathcal{F}_n\}$ . Then, for  $p \geq 2$ , we have*

$$\mathbb{E} \left| \sum X_k \right|^p \leq K_p \left( \mathbb{E} \left( \sum \mathbb{E}(|X_k|^2 | \mathcal{F}_{k-1}) \right)^{p/2} + \mathbb{E} \sum |X_k|^p \right).$$

**Lemma 2.2** (*Lemma A.1 of Bai and Silverstein (1998)*). *For  $X = (X_1, \dots, X_n)'$  i.i.d. standardized (complex) entries,  $\mathbf{B}$   $n \times n$  Hermitian nonnegative definite matrix, we have, for any  $p \geq 1$ ,*

$$\mathbb{E} |\mathbf{X}^* \mathbf{B} \mathbf{X}|^p \leq K_p \left( (\text{tr} \mathbf{B})^p + r \mathbb{E} |X_1|^{2p} \text{tr} \mathbf{B}^p \right),$$

where  $K_p$  is a constant depending on  $p$  only.

**Lemma 2.3** (*Jin et al. (2014)*). *Assume:*

(a)  $\tau \geq 1$  is a fixed integer.

(b)  $\mathbf{e}_k = (\varepsilon_{1k}, \dots, \varepsilon_{nk})'$ ,  $k = 1, 2, \dots, T + \tau$ , are  $n$ -dimensional vectors of independent standard complex components with  $\sup_{1 \leq i \leq n, 1 \leq t \leq T + \tau} \mathbb{E} |\varepsilon_{it}|^{2+\delta} \leq M < \infty$  for some  $\delta \in (0, 2]$ , and for any  $\eta > 0$ ,

$$\frac{1}{\eta^{2+\delta} n T} \sum_{i=1}^n \sum_{t=1}^{T+\tau} \mathbb{E} (|\varepsilon_{it}|^{2+\delta} I(|\varepsilon_{it}| \geq \eta T^{1/(2+\delta)})) = o(1). \quad (2.1)$$

(c)  $n/(T + \tau) \rightarrow c > 0$  as  $n, T \rightarrow \infty$ .

(d)  $\mathbf{M}_n(\tau) = \sum_{k=1}^T (\gamma_k \gamma_{k+\tau}^* + \gamma_{k+\tau} \gamma_k^*)$ , where  $\gamma_k = \frac{1}{\sqrt{2T}} \mathbf{e}_k$ .

Then as  $n, T \rightarrow \infty$ ,  $F^{\mathbf{M}_n(\tau)} \xrightarrow{D} F_c$  a.s. and  $F_c$  has a density function given by

$$\phi_c(x) = \frac{1}{2c\pi} \sqrt{\frac{y_0^2}{1+y_0} - \left( \frac{1-c}{|x|} + \frac{1}{\sqrt{1+y_0}} \right)^2}, \quad |x| \leq a,$$

where

$$a = \begin{cases} \frac{(1-c)\sqrt{1+y_1}}{y_1-1}, & c \neq 1, \\ 2, & c = 1, \end{cases}$$

$y_0$  is the largest real root of the equation:  $y^3 - \frac{(1-c)^2-x^2}{x^2}y^2 - \frac{4}{x^2}y - \frac{4}{x^2} = 0$  and  $y_1$  is the only real root of the equation:

$$((1-c)^2 - 1)y^3 + y^2 + y - 1 = 0 \quad (2.2)$$

such that  $y_1 > 1$  if  $c < 1$  and  $y_1 \in (0, 1)$  if  $c > 1$ . Further, if  $c > 1$ , then  $F_c$  has a point mass  $1 - 1/c$  at the origin. Note that as long as  $\tau \geq 1$ ,  $F_c$  does not depend on  $\tau$ .

**Lemma 2.4** (Bai and Wang (2015)). *Theorem 2.3 still holds with the  $2 + \delta$  moment condition weakened to 2nd moment.*

**Lemma 2.5** (Wang et al. (2015)). *Assume:*

(a)  $\tau \geq 1$  is a fixed integer.

(b)  $\mathbf{e}_k = (\varepsilon_{1k}, \dots, \varepsilon_{nk})'$ ,  $k = 1, 2, \dots, T + \tau$ , are  $n$ -vectors of independent standard complex components with  $\sup_{i,t} E|\varepsilon_{it}|^4 \leq M$  for some  $M > 0$ .

(c) There exist  $K > 0$  and a random variable  $X$  with finite fourth order moment such that, for any  $x > 0$ , for all  $n, T$

$$\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^{T+\tau} P(|\varepsilon_{it}| > x) \leq KP(|X| > x). \quad (2.3)$$

(d)  $c_n \equiv n/T \rightarrow c > 0$  as  $n \rightarrow \infty$ .

(e)  $\mathbf{M}_n = \sum_{k=1}^T (\gamma_k \gamma_{k+\tau}^* + \gamma_{k+\tau} \gamma_k^*)$ , where  $\gamma_k = \frac{1}{\sqrt{2T}} \mathbf{e}_k$ .

(f) The interval  $[a, b]$  lies outside the support of  $F_c$ , where  $F_c$  is defined as in Lemma 2.3.

Then  $P(\text{no eigenvalues of } \mathbf{M}_n \text{ appear in } [a, b] \text{ for all large } n) = 1$ .

### 3 Truncation, centralization and standardization of variables

As proved in Wang et al.(2015), we may assume that the  $\varepsilon_{ij}$ 's satisfy the conditions that

$$|\varepsilon_{ij}| \leq C, \mathbb{E}\varepsilon_{ij} = 0, \mathbb{E}|\varepsilon_{ij}|^2 = 1, \mathbb{E}|\varepsilon_{ij}|^4 < M \quad (3.4)$$

for some  $C, M > 0$ .

For the truncation of variables in  $\mathbf{F}^\tau$ , first note that for a random variable  $X$  with  $\mathbb{E}|X|^4 < \infty$ , we have  $\sum_{\ell=1}^{\infty} 2^\ell \mathbb{P}(|X| > 2^{\ell/4}) < \infty$ . Given  $\mathbb{E}|\mathbf{F}_{ij}^\tau|^4 < \infty, i = 1, \dots, k(q+1), j = 1, \dots, T$ ,

$$\text{define } \hat{\mathbf{F}}_{ij}^\tau = \begin{cases} \mathbf{F}_{ij}^\tau, & |\mathbf{F}_{ij}^\tau| < T^{1/4}, \\ 0, & \text{otherwise,} \end{cases} \quad \hat{\mathbf{F}}^\tau = (\hat{\mathbf{F}}_{ij}^\tau) \text{ and}$$

$$\hat{\Phi}_n(\tau) = \frac{1}{2T} [(\Lambda \hat{\mathbf{F}}^\tau + \mathbf{e}^\tau)(\Lambda \hat{\mathbf{F}}^0 + \mathbf{e}^0)^* + (\Lambda \hat{\mathbf{F}}^0 + \mathbf{e}^0)(\Lambda \hat{\mathbf{F}}^\tau + \mathbf{e}^\tau)^*].$$

Then we have

$$\begin{aligned} & \mathbb{P}(\Phi_n(\tau) \neq \hat{\Phi}_n(\tau), i.o.) \\ &= \mathbb{P}(\mathbf{F}^\tau \neq \hat{\mathbf{F}}^\tau, i.o.) \\ &= \mathbb{P}\left(\bigcap_{L=1}^{\infty} \bigcup_{T=L}^{\infty} \bigcup_{\substack{i \leq k(q+1) \\ j \leq T}} \{|\mathbf{F}_{ij}^\tau| \geq T^{1/4}\}\right) \\ &\leq \lim_{L \rightarrow \infty} \sum_{\ell=L}^{\infty} \mathbb{P}\left(\bigcup_{T=2^{\ell+1}}^{2^{\ell+1}} \bigcup_{\substack{i \leq k(q+1) \\ j \leq 2^{\ell+1}}} \{|\mathbf{F}_{ij}^\tau| \geq 2^{\ell/4}\}\right) \\ &\leq \lim_{L \rightarrow \infty} \sum_{\ell=L}^{\infty} \mathbb{P}\left(\bigcup_{\substack{i \leq k(q+1) \\ j \leq 2^{\ell+1}}} \{|\mathbf{F}_{ij}^\tau| \geq 2^{\ell/4}\}\right) \\ &\leq k(q+1) \lim_{L \rightarrow \infty} \sum_{\ell=L}^{\infty} 2^{\ell+1} \mathbb{P}(|\mathbf{F}_{11}^\tau| \geq 2^{\ell/4}) \\ &\rightarrow 0. \end{aligned}$$

This completes the proof of truncation. Centralization and standardization can be justified in the

same way as in Appendix A of Wang et al. (2015). In what follows, we may assume that

$$|\mathbf{F}_{ij}^\tau| < T^{1/4}, \quad \mathbf{E}\mathbf{F}_{ij}^\tau = 0, \quad \mathbf{E}|\mathbf{F}_{ij}^\tau|^2 = 1, \quad \mathbf{E}|\mathbf{F}_{ij}^\tau|^4 < M$$

for some  $M > 0$ .

## 4 Estimation of $k(q+1)$

In this section, we will estimate  $k(q+1)$  by an investigation of the limiting properties of eigenvalues of  $\Phi_n(0)$ . For simplicity, rewrite  $\Phi_n(0) = \Phi(0)$ ,  $\mathbf{F}^0 = \mathbf{F}$  and  $\mathbf{e}^0 = \mathbf{e}$ . With these notations, we have  $\Phi(0) = \frac{1}{T}(\Lambda\mathbf{F} + \mathbf{e})(\Lambda\mathbf{F} + \mathbf{e})^*$  and  $\mathbf{M}(0) = \frac{1}{T}\mathbf{e}\mathbf{e}^*$ . When  $\Lambda = \mathbf{0}$ ,  $\Phi(0)$  reduces to  $\mathbf{M}(0)$ , which is a standard sample covariance matrix and thus its ESD tends to the famous MP law (Marčenko and Pastur, 1967).

Suppose  $\ell$  is an eigenvalue of  $\Phi(0)$ , then we have

$$0 = \det |\ell\mathbf{I} - \Phi(0)| = \det \left| \ell\mathbf{I} - \mathbf{M}(0) - \frac{1}{T}\Lambda\mathbf{F}\mathbf{e}^* - \frac{1}{T}\mathbf{e}\mathbf{F}^*\Lambda^* - \frac{1}{T}\Lambda\mathbf{F}\mathbf{F}^*\Lambda^* \right|. \quad (4.1)$$

Let  $\mathbf{B} = (\mathbf{B}_1 : \mathbf{B}_2)$  be an  $n \times n$  orthogonal matrix such that  $\mathbf{B}_1 = \Lambda(\Lambda^*\Lambda)^{-1/2}$  and thus  $\Lambda^*\mathbf{B}_2 = \mathbf{0}_{k(q+1) \times (n-k(q+1))}$ . Then (4.1) is equivalent to

$$\det \begin{vmatrix} \ell\mathbf{I}_{k(q+1)} - \frac{1}{T}\mathbf{B}_1^*(\Lambda\mathbf{F} + \mathbf{e})(\mathbf{F}^*\Lambda^* + \mathbf{e}^*)\mathbf{B}_1 & -\frac{1}{T}\mathbf{B}_1^*(\Lambda\mathbf{F} + \mathbf{e})\mathbf{e}^*\mathbf{B}_2 \\ -\frac{1}{T}\mathbf{B}_2^*\mathbf{e}(\mathbf{F}^*\Lambda^* + \mathbf{e}^*)\mathbf{B}_1 & \ell\mathbf{I}_{n-k(q+1)} - \frac{1}{T}\mathbf{B}_2^*\mathbf{e}\mathbf{e}^*\mathbf{B}_2 \end{vmatrix} = 0 \quad (4.2)$$

If we further assume that  $\ell$  is not an eigenvalue of  $\frac{1}{T}\mathbf{B}_2^*\mathbf{e}\mathbf{e}^*\mathbf{B}_2$ , then we have

$$\det |\mathbf{I}_{k(q+1)} - \frac{1}{T}\mathbf{B}_1^*(\Lambda\mathbf{F} + \mathbf{e})\mathbf{D}^{-1}(\ell)(\mathbf{F}^*\Lambda^* + \mathbf{e}^*)\mathbf{B}_1| = 0, \quad (4.3)$$

where  $\mathbf{D}(\ell) = \ell\mathbf{I}_T - \frac{1}{T}\mathbf{e}^*\mathbf{B}_2\mathbf{B}_2^*\mathbf{e}$ . Denote  $\mathbf{H}(\ell) = \ell\mathbf{I}_T - \frac{1}{T}\mathbf{e}^*\mathbf{e}$ , then we obtain

$$\frac{1}{T}\mathbf{B}_1^*\mathbf{e}\mathbf{D}^{-1}(\ell)\mathbf{e}^*\mathbf{B}_1 = \left( \mathbf{I} + \frac{1}{T}\mathbf{B}_1^*\mathbf{e}\mathbf{H}^{-1}(\ell)\mathbf{e}^*\mathbf{B}_1 \right)^{-1} \frac{1}{T}\mathbf{B}_1^*\mathbf{e}\mathbf{H}^{-1}(\ell)\mathbf{e}^*\mathbf{B}_1. \quad (4.4)$$

Next, we have

$$\begin{aligned} \frac{1}{T}\mathbf{B}_1^*\mathbf{e}\mathbf{H}^{-1}(\ell)\mathbf{e}^*\mathbf{B}_1 &= \frac{1}{T}\mathbf{B}_1^* \left( -T\mathbf{I} + \ell T(\ell\mathbf{I}_n - \mathbf{M}(0))^{-1} \right) \mathbf{B}_1 \\ &= -\mathbf{I}_{k(q+1)} + \ell\mathbf{B}_1^*(\ell\mathbf{I}_n - \mathbf{M}(0))^{-1}\mathbf{B}_1. \end{aligned} \quad (4.5)$$

Substitute (4.5) back to (4.4), and we have

$$\begin{aligned} & \frac{1}{T} \mathbf{B}_1^* \mathbf{e} \mathbf{D}^{-1}(\ell) \mathbf{e}^* \mathbf{B}_1 \\ &= \mathbf{I}_{k(q+1)} - \left( \ell \mathbf{B}_1^* (\ell \mathbf{I}_n - \mathbf{M}(0))^{-1} \mathbf{B}_1 \right)^{-1} = \mathbf{I}_{k(q+1)} + \frac{1}{\ell} \left( \mathbf{B}_1^* (\mathbf{M}(0) - \ell \mathbf{I}_n)^{-1} \mathbf{B}_1 \right)^{-1} \end{aligned}$$

Write  $\mathbf{B}_1 = (\mathbf{b}_1, \dots, \mathbf{b}_{k(q+1)})$ , then we have  $\|\mathbf{b}_i\| = 1$ . By Lemma 6 in Bai, Liu and Wong (2011), we have

$$\mathbf{b}_i^* (\mathbf{M}(0) - \ell \mathbf{I}_n)^{-1} \mathbf{b}_i \rightarrow m, \quad a.s.$$

and for  $i \neq j$

$$\mathbf{b}_i^* (\mathbf{M}(0) - \ell \mathbf{I}_n)^{-1} \mathbf{b}_j \rightarrow 0, \quad a.s.,$$

where  $m = m(\ell) = \lim_{n \rightarrow \infty} \frac{1}{n} \text{tr}(\mathbf{M}(0) - \ell \mathbf{I}_n)^{-1}$  is the Stieltjes transform of the sample covariance with ratio index  $c = \lim_{n \rightarrow \infty} \frac{n}{T}$ .

By Lemma 3.11 of Bai and Silverstein (2010), we have  $m$  satisfying

$$m(\ell) = \frac{1 - c - \ell + \sqrt{(1 - \ell - c)^2 - 4\ell c}}{2c\ell}. \quad (4.6)$$

Therefore, we obtain

$$\frac{1}{T} \mathbf{B}_1^* \mathbf{e} \mathbf{D}^{-1}(\ell) \mathbf{e}^* \mathbf{B}_1 \rightarrow \left(1 + \frac{1}{\ell m}\right) \mathbf{I}_{k(q+1)}.$$

Next, we want to show, with probability 1 that

$$\frac{1}{T} \mathbf{B}_1^* \mathbf{\Lambda} \mathbf{F} \mathbf{D}^{-1}(\ell) \mathbf{e}^* \mathbf{B}_1 \rightarrow \mathbf{0}$$

and

$$\frac{1}{T} \mathbf{B}_1^* \mathbf{e} \mathbf{D}^{-1}(\ell) \mathbf{F}^* \mathbf{\Lambda}^* \mathbf{B}_1 \rightarrow \mathbf{0}.$$

Note that

$$\begin{aligned} & \frac{1}{T} \mathbf{B}_1^* \mathbf{\Lambda} \mathbf{F} \mathbf{D}^{-1}(\ell) \mathbf{e}^* \mathbf{B}_1 \\ &= \frac{1}{T} \left( \mathbf{I} + \frac{1}{T} \mathbf{B}_1^* \mathbf{e} \mathbf{H}^{-1}(\ell) \mathbf{e}^* \mathbf{B}_1 \right)^{-1} \mathbf{B}_1^* \mathbf{\Lambda} \mathbf{F} \mathbf{H}^{-1}(\ell) \mathbf{e}^* \mathbf{B}_1 \\ &= \frac{1}{\ell T} \left( \mathbf{B}_1^* (\mathbf{M}(0) - \ell \mathbf{I}_N)^{-1} \mathbf{B}_1 \right)^{-1} \mathbf{B}_1^* \mathbf{\Lambda} \mathbf{F} \mathbf{H}^{-1}(\ell) \mathbf{e}^* \mathbf{B}_1 \\ &= \frac{1}{\ell T} \left( \mathbf{B}_1^* (\mathbf{M}(0) - \ell \mathbf{I}_N)^{-1} \mathbf{B}_1 \right)^{-1} \mathbf{B}_1^* \mathbf{\Lambda} \mathbf{F} \left( \ell \mathbf{I}_T - \frac{1}{T} \mathbf{e}^* \mathbf{e} \right)^{-1} \mathbf{e}^* \mathbf{B}_1. \end{aligned}$$



Recall  $\mathbf{M}(0) = \frac{1}{T}\mathbf{e}\mathbf{e}^*$ . Fix  $\delta > 0$  and let event  $\mathcal{A} = \{\lambda_{\max}(\mathbf{M}(0)) \leq (1 + \sqrt{c})^2 + \delta\}$  and  $\mathcal{A}^c$  be the complement. By Theorem 5.9 of Bai and Silverstein (2010), we have  $\mathbb{P}(\mathcal{A}^c) = o(n^{-t})$  for any  $t > 0$ .

Suppose  $\ell$  is an eigenvalue of  $\Phi(0)$  larger than  $(1 + \sqrt{c})^2 + 2\delta$ . By the fact that  $\mathbf{M}(0)$  and  $\frac{1}{T}\mathbf{e}^*\mathbf{e}$  have the same set of nonzero eigenvalues, we have, under  $\mathcal{A}$ , that  $\|\ell\mathbf{I}_T - \frac{1}{T}\mathbf{e}^*\mathbf{e}\|_o \geq \ell - \|\frac{1}{T}\mathbf{e}^*\mathbf{e}\|_o \geq \delta > 0$ , and hence  $\|(\ell\mathbf{I}_T - \frac{1}{T}\mathbf{e}^*\mathbf{e})^{-1}\|_o \leq \frac{1}{\delta}$ .

Therefore, for any  $\varepsilon > 0$ , we have

$$\begin{aligned} & \mathbb{P}(\|\frac{1}{T}\mathbf{F}(\ell\mathbf{I}_T - \frac{1}{T}\mathbf{e}^*\mathbf{e})^{-1}\mathbf{e}^*\mathbf{B}_1\|_K \geq \varepsilon) \\ &= \mathbb{E}\left(\mathbb{P}(\|\frac{1}{T}\mathbf{F}(\ell\mathbf{I}_T - \frac{1}{T}\mathbf{e}^*\mathbf{e})^{-1}\mathbf{e}^*\mathbf{B}_1\|_K \geq \varepsilon) \middle| \mathbf{e}\right) \\ &\leq \mathbb{E}\left(\mathbb{P}(\|\frac{1}{T}\mathbf{F}(\ell\mathbf{I}_T - \frac{1}{T}\mathbf{e}^*\mathbf{e})^{-1}\mathbf{e}^*\mathbf{B}_1\|_K \geq \varepsilon) \middle| \mathbf{e}, \mathcal{A}\right) + \mathbb{P}(\mathcal{A}^c). \end{aligned}$$

Write  $\mathbf{F} = (\tilde{\mathbf{F}}_1, \dots, \tilde{\mathbf{F}}_{k(q+1)})'$ . For the first term, by Lemma 2.2, we have

$$\begin{aligned} & \mathbb{E}\left(\mathbb{P}(\|\frac{1}{T}\mathbf{F}(\ell\mathbf{I}_T - \frac{1}{T}\mathbf{e}^*\mathbf{e})^{-1}\mathbf{e}^*\mathbf{B}_1\|_K \geq \varepsilon) \middle| \mathbf{e}, \mathcal{A}\right) \\ &\leq \frac{1}{\varepsilon^{4r}T^{4r}}\mathbb{E}\mathbb{E}\left(\|\mathbf{F}(\ell\mathbf{I}_T - \frac{1}{T}\mathbf{e}^*\mathbf{e})^{-1}\mathbf{e}^*\mathbf{B}_1\|_K^{4r} \middle| \mathbf{e}, \mathcal{A}\right) \\ &\leq \frac{1}{\varepsilon^{4r}T^{4r}}\mathbb{E}\mathbb{E}\left[\left(\text{tr}\mathbf{F}(\ell\mathbf{I}_T - \frac{1}{T}\mathbf{e}^*\mathbf{e})^{-1}\mathbf{e}^*\mathbf{B}_1\mathbf{B}_1^*\mathbf{e}(\ell\mathbf{I}_T - \frac{1}{T}\mathbf{e}^*\mathbf{e})^{-1}\mathbf{F}^*\right)^{2r} \middle| \mathbf{e}, \mathcal{A}\right] \\ &= \frac{1}{\varepsilon^{4r}T^{4r}}\mathbb{E}\mathbb{E}\left[\left(\sum_{i=1}^{k(q+1)} \tilde{\mathbf{F}}_i(\ell\mathbf{I}_T - \frac{1}{T}\mathbf{e}^*\mathbf{e})^{-1}\mathbf{e}^*\mathbf{B}_1\mathbf{B}_1^*\mathbf{e}(\ell\mathbf{I}_T - \frac{1}{T}\mathbf{e}^*\mathbf{e})^{-1}\tilde{\mathbf{F}}_i^*\right)^{2r} \middle| \mathbf{e}, \mathcal{A}\right] \\ &\leq \frac{[k(q+1)]^{2r-1}}{\varepsilon^{4r}T^{4r}}\mathbb{E}\left[\sum_{i=1}^{k(q+1)} \mathbb{E}\left(\tilde{\mathbf{F}}_i(\ell\mathbf{I}_T - \frac{1}{T}\mathbf{e}^*\mathbf{e})^{-1}\mathbf{e}^*\mathbf{B}_1\mathbf{B}_1^*\mathbf{e}(\ell\mathbf{I}_T - \frac{1}{T}\mathbf{e}^*\mathbf{e})^{-1}\tilde{\mathbf{F}}_i^*\right)^{2r} \middle| \mathbf{e}, \mathcal{A}\right] \\ &\leq \frac{K_{2r}[k(q+1)]^{2r-1}}{\varepsilon^{4r}T^{4r}}\mathbb{E}\left[\sum_{i=1}^{k(q+1)} \left(\left[\text{tr}(\ell\mathbf{I}_T - \frac{1}{T}\mathbf{e}^*\mathbf{e})^{-1}\mathbf{e}^*\mathbf{B}_1\mathbf{B}_1^*\mathbf{e}(\ell\mathbf{I}_T - \frac{1}{T}\mathbf{e}^*\mathbf{e})^{-1}\right]^{2r} + \right. \right. \\ &\quad \left. \left. \mathbb{E}|\mathbf{F}_{11}|^{4r}\text{tr}[(\ell\mathbf{I}_T - \frac{1}{T}\mathbf{e}^*\mathbf{e})^{-1}\mathbf{e}^*\mathbf{B}_1\mathbf{B}_1^*\mathbf{e}(\ell\mathbf{I}_T - \frac{1}{T}\mathbf{e}^*\mathbf{e})^{-1}]^{2r}\right) \middle| \mathcal{A}\right] \\ &\leq \frac{K_{2r}[k(q+1)]^{2r}}{\delta^{4r}\varepsilon^{4r}T^{4r}}\mathbb{E}\left[\left(\text{tr}^*\mathbf{B}_1\mathbf{B}_1^*\mathbf{e}\right)^{2r} + \mathbb{E}|\mathbf{F}_{11}|^{4r}\text{tr}(\mathbf{e}^*\mathbf{B}_1\mathbf{B}_1^*\mathbf{e})^{2r} \middle| \mathcal{A}\right] \\ &\leq \frac{K_{2r}[k(q+1)]^{2r}[(1 + \sqrt{c})^2 + \delta]^{2r}}{\delta^{4r}\varepsilon^{4r}T^{4r}}\mathbb{E}\left(\left(\text{tr}\mathbf{B}_1^*\mathbf{B}_1\right)^{2r} + \mathbb{E}|\mathbf{F}_{11}|^{4r}\text{tr}(\mathbf{B}_1^*\mathbf{B}_1)^{2r}\right) \\ &\leq \frac{K_{2r}[k(q+1)]^{2r}[(1 + \sqrt{c})^2 + \delta]^{2r}}{\delta^{4r}\varepsilon^{4r}T^{4r}}\left([k(q+1)]^{2r} + \mathbb{E}|\mathbf{F}_{11}|^4 T^{\frac{4r-4}{4}} k(q+1)\right) \end{aligned}$$

which is summable for  $r \geq 1$ .

Hence, we have shown with probability 1 that

$$\frac{1}{T} \mathbf{B}_1^* \mathbf{\Lambda} \mathbf{F} \mathbf{D}^{-1}(\ell) \mathbf{e}^* \mathbf{B}_1 \rightarrow \mathbf{0}.$$

Similarly, we have with probability 1 that

$$\frac{1}{T} \mathbf{B}_1^* \mathbf{e} \mathbf{D}^{-1}(\ell) \mathbf{F}^* \mathbf{\Lambda}^* \mathbf{B}_1 \rightarrow \mathbf{0}.$$

Therefore, substituting into (4.3), we have

$$\det \left| \frac{1}{T} \mathbf{B}_1^* \mathbf{\Lambda} \mathbf{F} \mathbf{D}^{-1}(\ell) \mathbf{F}^* \mathbf{\Lambda}^* \mathbf{B}_1 + \frac{1}{\ell m(\ell)} \mathbf{I}_{k(q+1)} \right| \rightarrow 0. \quad (4.7)$$

Using Bai, Liu and Wong (2011) again, we have the diagonal elements of the matrix  $T^{-1} \mathbf{F} \mathbf{D}^{-1}(\ell) \mathbf{F}^*$  tend to  $-\underline{m}(\ell)$  and the off diagonal elements tend to 0. Here  $\underline{m}(\ell)$  is the Stieltjes transform of the LSD of  $\frac{1}{T} \mathbf{e}^* \mathbf{e}$  and satisfies

$$\underline{m}(\ell) = -\frac{1-c}{\ell} + c m(\ell).$$

Thus, if  $\mathbf{\Lambda}^* \mathbf{\Lambda} \rightarrow \mathbf{Q}$ , then (4.7) can be further simplified as

$$\det \left| -\mathbf{Q} \underline{m}(\ell) + \frac{1}{\ell m(\ell)} \mathbf{I}_{k(q+1)} \right| = 0. \quad (4.8)$$

If  $\alpha$  is an eigenvalue of  $\mathbf{Q}$ , and there is an  $\ell$  belonging to the complement of the support of the LSD of  $\mathbf{M}(0)$  such that  $\alpha = \frac{1}{\ell m(\ell) \underline{m}(\ell)}$ , then  $\ell$  is a solution of (4.8).

From (4.6), we have

$$c \ell m^2(\ell) - (1 - c - \ell) m(\ell) + 1 = 0,$$

which implies

$$\begin{aligned} \ell m(\ell) \underline{m}(\ell) &= \ell m(\ell) \left( -\frac{1-c}{\ell} + c m(\ell) \right) \\ &= -(1-c) m(\ell) + c \ell m^2(\ell) \\ &= -(1-c) m(\ell) + (1-c-\ell) m(\ell) - 1 \\ &= -\ell m(\ell) - 1 \\ &= -\frac{1-c-\ell + \sqrt{(1-\ell-c)^2 - 4\ell c}}{2c} - 1 \\ &=: g(\ell). \end{aligned}$$

It is easy to verify that  $g'(\ell) < 0$ , implying that  $\ell m(\ell) \underline{m}(\ell)$  is decreasing. Also note that  $\ell m(\ell) \underline{m}(\ell) = \frac{1}{\sqrt{c}}$  when  $\ell = (1 + \sqrt{c})^2$ . Therefore, if  $\alpha = \frac{1}{\ell m(\ell) \underline{m}(\ell)} > \sqrt{c}$ , then we have  $\ell > (1 + \sqrt{c})^2$ . This recovers the result of Baik and Silverstein (2006). Note that  $[(1 - \sqrt{c})^2, (1 + \sqrt{c})^2]$  is the support of the MP law. Hence, if all the eigenvalues of  $\mathbf{Q}$  are greater than  $\sqrt{c}$ , we have  $k(q + 1)$  sample eigenvalues of  $\mathbf{M}_n(0)$  goes outside the right boundary of the support of the MP law. In this way,  $k(q + 1)$  can be estimated.

*Remark 4.1* For factor models, the loading matrix is unknown. This, however, is not a concern in our estimation because compared with the noise matrix, the loading matrix is denominating, making the condition easily satisfied that all the eigenvalues of  $\mathbf{Q}$  are greater than  $\sqrt{c}$ .

## 5 Estimation of $q$

Next, we want to split  $k$  and  $q$ . Let  $\tau \geq 1$  be given and assume that  $\ell$  is an eigenvalue of  $\Phi_n(\tau)$ . For simplicity, write  $\mathbf{M}_n(\tau) = \mathbf{M}$  and for  $t = 1, 2, \dots, T$ , define  $\mathbf{F}_t = (\mathbf{f}_t, \mathbf{f}_{t-1}, \dots, \mathbf{f}_{t-q})'$  such that  $\mathbf{R}_t = \Lambda \mathbf{F}_t + \mathbf{e}_t$ . Then we have

$$\begin{aligned} 0 &= \det |\ell \mathbf{I} - \Phi_n(\tau)| \\ &= \det \left| \ell \mathbf{I} - \mathbf{M} - \frac{1}{2T} \sum_{j=1}^T \left( \Lambda \mathbf{F}_j \mathbf{F}_{j+\tau}^* \Lambda^* + \Lambda \mathbf{F}_{j+\tau}^* \mathbf{F}_j \Lambda^* \right. \right. \\ &\quad \left. \left. + \mathbf{e}_j \mathbf{F}_{j+\tau}^* \Lambda^* + \Lambda \mathbf{F}_{j+\tau}^* \mathbf{e}_j^* + \mathbf{e}_{j+\tau}^* \mathbf{F}_j \Lambda^* + \Lambda \mathbf{F}_j \mathbf{e}_{j+\tau}^* \right) \right|. \end{aligned} \quad (5.1)$$

Define  $\mathbf{B}, \mathbf{B}_1$  and  $\mathbf{B}_2$  the same as in the last section. Multiplying  $\mathbf{B}^*$  from left and  $\mathbf{B}$  from right to the above matrix and by  $\Lambda^* \mathbf{B}_2 = \mathbf{0}_{k(q+1) \times (n-k(q+1))}$ , we have (5.1) equivalent to

$$0 = \det \begin{vmatrix} \ell \mathbf{I}_{k(q+1)} - \mathbf{S}_{11} & -\mathbf{S}_{12} \\ -\mathbf{S}_{21} & \ell \mathbf{I}_{n-k(q+1)} - \mathbf{S}_{22} \end{vmatrix} = \det |\ell \mathbf{I} - \mathbf{S}_{22}| \det |\ell \mathbf{I} - \mathbf{K}_n(\ell)|,$$

where

$$\begin{aligned}
\mathbf{S}_{11} &= \frac{1}{2T} \sum_{j=1}^T \mathbf{B}_1^* [(\boldsymbol{\Lambda} \mathbf{F}_j + \mathbf{e}_j)(\boldsymbol{\Lambda} \mathbf{F}_{j+\tau} + \mathbf{e}_{j+\tau})^* + (\boldsymbol{\Lambda} \mathbf{F}_{j+\tau} + \mathbf{e}_{j+\tau})(\boldsymbol{\Lambda} \mathbf{F}_j + \mathbf{e}_j)^*] \mathbf{B}_1 \\
\mathbf{S}_{12} &= \frac{1}{2T} \sum_{j=1}^T \mathbf{B}_1^* (\boldsymbol{\Lambda} \mathbf{F}_j \mathbf{e}_{j+\tau}^* + \boldsymbol{\Lambda} \mathbf{F}_{j+\tau} \mathbf{e}_j^*) \mathbf{B}_2 + \mathbf{B}_1^* \mathbf{M} \mathbf{B}_2 \\
\mathbf{S}_{21} &= \mathbf{S}_{12}^* \\
\mathbf{S}_{22} &= \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2 \\
\mathbf{K}_n(\ell) &= \mathbf{S}_{11} + \mathbf{S}_{12}(\ell \mathbf{I}_{n-k(q+1)} - \mathbf{S}_{22})^{-1} \mathbf{S}_{21}.
\end{aligned}$$

Therefore, if  $\ell$  is not an eigenvalue of  $\mathbf{S}_{22}$ , by the factorization above,  $\ell$  must be an eigenvalue of  $\mathbf{K}_n(\ell)$ , i.e.  $\det |\mathbf{K}_n(\ell) - \ell \mathbf{I}| = 0$ .

Denote  $\mathbf{W} = \frac{1}{2T} \sum_{j=1}^T (\mathbf{F}_j \mathbf{e}_{j+\tau}^* + \mathbf{F}_{j+\tau} \mathbf{e}_j^*)$ . By the assumptions of  $\mathbf{e}_t$ 's and  $\mathbf{F}_t$ 's, the random vector  $\{\mathbf{F}_j \mathbf{e}_{j+\tau}^* + \mathbf{F}_{j+\tau} \mathbf{e}_j^*, j \geq 1\}$  is  $(q+1)$ -dependent (see Page 224, Chung 2001). It then follows with probability 1 that

$$\begin{aligned}
\mathbf{W} \mathbf{B}_1 &= o(\mathbf{1}) \\
\mathbf{B}_1^* \mathbf{W}^* &= o(\mathbf{1}) \\
\frac{1}{2T} \sum_{j=1}^T (\mathbf{F}_j \mathbf{F}_{j+\tau}^* + \mathbf{F}_{j+\tau} \mathbf{F}_j^*) &= \mathbf{H}(\tau) + o(\mathbf{1}),
\end{aligned}$$

where

$$\mathbf{H}(\tau) = \begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & 0 & 1 & \vdots \\ 1 & 0 & \ddots & 0 & 1 \\ \vdots & 1 & 0 & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \end{pmatrix},$$

is of dimension  $k(q+1) \times k(q+1)$  with two bands of 1's of  $k\tau$ -distance from the main diagonal.

Therefore, we have a.s.

$$\begin{aligned} \mathbf{S}_{11} &= \mathbf{B}_1^* \Lambda \mathbf{H}(\tau) \Lambda^* \mathbf{B}_1 + \mathbf{B}_1^* \mathbf{M} \mathbf{B}_1 + o(\mathbf{1}) \\ \mathbf{S}_{12} &= \mathbf{B}_1^* (\mathbf{M} + \Lambda \mathbf{W}) \mathbf{B}_2 \\ \mathbf{S}_{21} &= \mathbf{B}_2^* (\mathbf{M} + \mathbf{W}^* \Lambda^*) \mathbf{B}_1. \end{aligned}$$

Subsequently, we have a.s.

$$\mathbf{K}_n(\ell) = \mathbf{B}_1^* \Lambda \mathbf{H}(\tau) \Lambda^* \mathbf{B}_1 + \mathbf{B}_1^* \mathbf{M} \mathbf{B}_1 + \mathbf{B}_1^* (\mathbf{M} + \Lambda \mathbf{W}) \mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* (\mathbf{M} + \mathbf{W}^* \Lambda^*) \mathbf{B}_1 + o(\mathbf{1}).$$

Note that

$$\begin{aligned} & \mathbf{B}_1^* \mathbf{M} \mathbf{B}_1 + \mathbf{B}_1^* \mathbf{M} \mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* \mathbf{M} \mathbf{B}_1 \\ &= \mathbf{B}_1^* \mathbf{M} \mathbf{B}_1 + \mathbf{B}_1^* \mathbf{M} \frac{1}{\ell} \mathbf{B}_2 \mathbf{B}_2^* \mathbf{M} \left( \mathbf{I} - \frac{1}{\ell} \mathbf{B}_2 \mathbf{B}_2^* \mathbf{M} \right)^{-1} \mathbf{B}_1 \\ &= \mathbf{B}_1^* \mathbf{M} \left( \mathbf{I} - \frac{1}{\ell} \mathbf{B}_2 \mathbf{B}_2^* \mathbf{M} \right)^{-1} \mathbf{B}_1 \\ &= \ell \mathbf{B}_1^* \mathbf{M} (\ell \mathbf{I} - \mathbf{M} + \mathbf{B}_1 \mathbf{B}_1^* \mathbf{M})^{-1} \mathbf{B}_1 \\ &= \ell \mathbf{I} - \ell \left( \mathbf{I} + \mathbf{B}_1^* \mathbf{M} (\ell \mathbf{I} - \mathbf{M})^{-1} \mathbf{B}_1 \right)^{-1} \\ &= \ell \mathbf{I} - \left( \mathbf{B}_1^* (\ell \mathbf{I} - \mathbf{M})^{-1} \mathbf{B}_1 \right)^{-1} \end{aligned}$$

and

$$\begin{aligned}
& \mathbf{W}\mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* \mathbf{W}^* \\
&= \frac{1}{2T} \sum_{i,j=1}^T (\mathbf{F}_i \gamma_{i+\tau}^* + \mathbf{F}_{i+\tau} \gamma_i^*) \mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* (\gamma_{j+\tau} \mathbf{F}_j^* + \gamma_j \mathbf{F}_{j+\tau}^*) \\
&= \frac{1}{2T} \sum_{j=1}^T \left[ \mathbf{F}_j \gamma_{j+\tau}^* \mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* \gamma_{j+\tau} \mathbf{F}_j^* + \mathbf{F}_{j+\tau} \gamma_j^* \mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* \gamma_j \mathbf{F}_{j+\tau}^* \right] + \\
&\quad \frac{1}{2T} \sum_{j=1}^T \left[ \mathbf{F}_j \gamma_{j+\tau}^* \mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* \gamma_j \mathbf{F}_{j+\tau}^* + \mathbf{F}_{j+\tau} \gamma_j^* \mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* \gamma_{j+\tau} \mathbf{F}_j^* \right] + \\
&\quad \frac{1}{2T} \sum_{j=1}^T (\mathbf{F}_{j+\tau} \gamma_{j+2\tau}^* + \mathbf{F}_{j+2\tau} \gamma_{j+\tau}^*) \mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* (\gamma_{j+\tau} \mathbf{F}_j^* + \gamma_j \mathbf{F}_{j+\tau}^*) + \\
&\quad \frac{1}{2T} \sum_{j=1}^T (\mathbf{F}_{j-\tau} \gamma_j^* + \mathbf{F}_j \gamma_{j-\tau}^*) \mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* (\gamma_{j+\tau} \mathbf{F}_j^* + \gamma_j \mathbf{F}_{j+\tau}^*) + \\
&\quad \frac{1}{2T} \sum_{\substack{i,j=1 \\ i \neq j, i \neq j \pm \tau}}^T (\mathbf{F}_i \gamma_{i+\tau}^* + \mathbf{F}_{i+\tau} \gamma_i^*) \mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* (\gamma_{j+\tau} \mathbf{F}_j^* + \gamma_j \mathbf{F}_{j+\tau}^*) \\
&=: \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 + \mathbf{P}_4 + \mathbf{P}_5.
\end{aligned}$$

Next, we give a lemma on the quadratic form of  $\gamma_j$ .

**Lemma 5.1** *Let  $i, j \in \mathbb{N}$  be given, we have almost surely and uniformly in  $i$  and  $j$  that*

$$\gamma_i^* \mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* \gamma_j \rightarrow \begin{cases} \frac{-\frac{cm}{2}}{1-\frac{c^2 m^2}{2x_1}} \left( \frac{-\frac{cm}{2}}{1-\frac{c^2 m^2}{4x_1}} \right)^p \equiv C_p, & i = j \pm p\tau \\ 0, & \text{otherwise.} \end{cases}$$

The proof of the lemma is postponed in the Appendix.

First, we have

$$\begin{aligned}
& \mathbf{E}(\mathbf{P}_1) \\
&= \mathbf{E}[\mathbf{E}(\mathbf{P}_1 | \gamma_1, \dots, \gamma_{T+\tau})] \\
&= \frac{1}{2T} \sum_{j=1}^T \mathbf{E} \left\{ \left[ \mathbf{F}_j \gamma_{j+\tau}^* \mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* \gamma_{j+\tau} \mathbf{F}_j^* + \right. \right. \\
&\quad \left. \left. \mathbf{F}_{j+\tau} \gamma_j^* \mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* \gamma_j \mathbf{F}_{j+\tau}^* \right] \middle| \gamma_1, \dots, \gamma_{T+\tau} \right\} \\
&= \frac{1}{2T} \text{Etr} \left[ \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* \sum_{j=1}^T (\gamma_{j+\tau} \gamma_{j+\tau}^* + \gamma_j \gamma_j^*) \mathbf{B}_2 \right] \mathbf{I}_{k(q+1)} \\
&= \frac{1}{T} \sum_{j=1}^T \mathbf{E} \gamma_j^* \mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* \gamma_j \mathbf{I}_{k(q+1)} \\
&= C_0 \mathbf{I}_{k(q+1)} + o(\mathbf{1}).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathbf{E}(P_2) &= \frac{1}{2} C_1 \mathbf{H}(\tau) + o(\mathbf{1}), \\
\mathbf{E}(P_3) &= C_1 \mathbf{H}_L(\tau) + \frac{1}{2} (C_0 \mathbf{H}_L(2\tau) + C_2 \mathbf{I}_{k(q+1)}) + o(\mathbf{1}), \\
\mathbf{E}(P_4) &= C_1 \mathbf{H}_U(\tau) + \frac{1}{2} (C_0 \mathbf{H}_U(2\tau) + C_2 \mathbf{I}_{k(q+1)}) + o(\mathbf{1}).
\end{aligned}$$

Here  $\mathbf{H}_L(\tau)$  and  $\mathbf{H}_U(\tau)$  denote the lower and upper part of  $\mathbf{H}(\tau)$  with the rest entries being 0.

Furthermore, denote  $\mathbf{H}_L(0) \equiv \mathbf{H}_U(0) \equiv \mathbf{I}_{k(q+1)}$  and hence  $\mathbf{H}(0) = \mathbf{H}_L(0) + \mathbf{H}_U(0) = 2\mathbf{I}_{k(q+1)}$ .

Consider  $i = j \pm p\tau$  for  $p = 0, 1, \dots, \lfloor \frac{q}{\tau} \rfloor$ .

When  $p = 0$ , we have  $\mathbf{E}(\mathbf{P}_1) + \mathbf{E}(\mathbf{P}_2) = \frac{1}{2} C_0 \mathbf{H}(0) + \frac{1}{2} C_1 \mathbf{H}(\tau) + o(\mathbf{1})$ .

When  $p = 1$ , we have  $\mathbf{E}(\mathbf{P}_3) + \mathbf{E}(\mathbf{P}_4) = \frac{1}{2} C_2 \mathbf{H}(0) + C_1 \mathbf{H}(\tau) + \frac{1}{2} C_0 \mathbf{H}(2\tau) + o(\mathbf{1})$ .

When  $p = 2$ , we have part of  $\mathbf{E}(\mathbf{P}_5)$  is  $\frac{1}{2} C_3 \mathbf{H}(\tau) + C_2 \mathbf{H}(2\tau) + \frac{1}{2} C_1 \mathbf{H}(3\tau) + o(\mathbf{1})$ .

$\vdots$

When  $p = \lfloor \frac{q}{\tau} \rfloor$ , we have part of  $E(\mathbf{P5})$  is

$$\begin{aligned} & \frac{1}{2}C_{\lfloor \frac{q}{\tau} \rfloor + 1} \mathbf{H}(\lfloor \frac{q}{\tau} \rfloor \tau - \tau) + C_{\lfloor \frac{q}{\tau} \rfloor} \mathbf{H}(\lfloor \frac{q}{\tau} \rfloor \tau) + \frac{1}{2}C_{\lfloor \frac{q}{\tau} \rfloor - 1} \mathbf{H}(\lfloor \frac{q}{\tau} \rfloor \tau + \tau) + o(\mathbf{1}) \\ &= \frac{1}{2}C_{\lfloor \frac{q}{\tau} \rfloor + 1} \mathbf{H}(\lfloor \frac{q}{\tau} \rfloor \tau - \tau) + C_{\lfloor \frac{q}{\tau} \rfloor} \mathbf{H}(\lfloor \frac{q}{\tau} \rfloor \tau) + o(\mathbf{1}). \end{aligned}$$

When  $p = \lfloor \frac{q}{\tau} \rfloor + 1$ , we have part of  $E(\mathbf{P5})$  is

$$\begin{aligned} & \frac{1}{2}C_{\lfloor \frac{q}{\tau} \rfloor} \mathbf{H}(\lfloor \frac{q}{\tau} \rfloor \tau + 2\tau) + C_{\lfloor \frac{q}{\tau} \rfloor + 1} \mathbf{H}(\lfloor \frac{q}{\tau} \rfloor \tau + \tau) + \frac{1}{2}C_{\lfloor \frac{q}{\tau} \rfloor + 2} \mathbf{H}(\lfloor \frac{q}{\tau} \rfloor \tau) + o(\mathbf{1}) \\ &= \frac{1}{2}C_{\lfloor \frac{q}{\tau} \rfloor + 2} \mathbf{H}(\lfloor \frac{q}{\tau} \rfloor \tau) + o(\mathbf{1}). \end{aligned}$$

Next, we want to show that  $\mathbf{P}_i \rightarrow E(\mathbf{P}_i)$  a.s. Since all the  $\mathbf{P}_i$ 's are of finite dimension, it suffices to show the a.s convergence entry-wise. Denote the  $(u, v)$ -entry of a matrix  $\mathbf{A}$  by  $\mathbf{A}_{(u,v)}$ . For  $i = 1$ , define  $\alpha_j = \gamma_j^* \mathbf{B}_2 (\ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2)^{-1} \mathbf{B}_2^* \gamma_j$ . Then for any positive integer  $s$ , applying Lemma 2.1, we have

$$\begin{aligned} & E(|\mathbf{P}_{1(i_1, i_2)} - \frac{\alpha_j}{2} \delta_{(i_1, i_2)} - \frac{\alpha_{j+\tau}}{2} \delta_{(i_1, i_2)}|^{2s}) \\ &= E\left\{ \frac{1}{2T} \sum_{j=1}^T \left[ \alpha_{j+\tau} (\mathbf{F}_j \mathbf{F}_j^*)_{(i_1, i_2)} + \alpha_j (\mathbf{F}_{j+\tau} \mathbf{F}_{j+\tau}^*)_{(i_1, i_2)} \right] - \frac{\alpha_j}{2} \delta_{(i_1, i_2)} - \frac{\alpha_{j+\tau}}{2} \delta_{(i_1, i_2)} \right\}^{2s} \\ &\leq 2^{2s-1} E\left[ \frac{1}{2T} \sum_{j=1}^T \alpha_{j+\tau} (\mathbf{F}_j \mathbf{F}_j^*)_{(i_1, i_2)} - \frac{\alpha_{j+\tau}}{2} \delta_{(i_1, i_2)} \right]^{2s} + \\ &\quad 2^{2s-1} E\left[ \frac{1}{2T} \sum_{j=1}^T \alpha_j (\mathbf{F}_{j+\tau} \mathbf{F}_{j+\tau}^*)_{(i_1, i_2)} - \frac{\alpha_j}{2} \delta_{(i_1, i_2)} \right]^{2s} \\ &= \frac{1}{2} E\left[ \frac{1}{T} \sum_{j=1}^T \alpha_{j+\tau} (\mathbf{F}_j \mathbf{F}_j^*)_{(i_1, i_2)} - \alpha_{j+\tau} \delta_{(i_1, i_2)} \right]^{2s} + \\ &\quad \frac{1}{2} E\left[ \frac{1}{T} \sum_{j=1}^T \alpha_j (\mathbf{F}_{j+\tau} \mathbf{F}_{j+\tau}^*)_{(i_1, i_2)} - \alpha_j \delta_{(i_1, i_2)} \right]^{2s} \\ &= \frac{1}{2} E\left\{ \frac{1}{T} \sum_{j=1}^T \mathbf{F}_j^* \left[ \alpha_{j+\tau}^{(i_1, i_2)} \right] \mathbf{F}_j - \text{tr} \left[ \alpha_{j+\tau}^{(i_1, i_2)} \right] \right\}^{2s} + \\ &\quad \frac{1}{2} E\left\{ \frac{1}{T} \sum_{j=1}^T \mathbf{F}_{j+\tau}^* \left[ \alpha_j^{(i_1, i_2)} \right] \mathbf{F}_{j+\tau} - \text{tr} \left[ \alpha_j^{(i_1, i_2)} \right] \right\}^{2s}. \end{aligned} \tag{5.2}$$



Here  $[a^{(u,v)}]$  denotes the matrix with the  $(u, v)$ -entry being  $a$  and 0 elsewhere. By the truncation of  $\varepsilon_{ij}$  and the fact that  $\|(\ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2)^{-1}\|_o \leq \eta^{-1}$  with  $\eta = \ell - d_c > 0$ , both  $|\alpha_j|$  and  $|\alpha_{j+\tau}|$  are bounded from above, say, by  $C$ . Also notice that  $|\mathbf{F}_{ij}^\tau| < T^{1/4}$  and  $\mathbb{E}|\mathbf{F}_{ij}^\tau|^4 < M$ . Similar to the proof of Lemma 9.1 in Bai and Silverstein (2010), we have

$$\begin{aligned} \mathbb{E} \left\{ \frac{1}{T} \sum_{j=1}^T \mathbf{F}_j^* [\alpha_{j+\tau}^{(i_1, i_2)}] \mathbf{F}_j - \text{tr} [\alpha_{j+\tau}^{(i_1, i_2)}] \right\}^{2s} &\leq \frac{C^{2s}}{T^s} \sum_{l=1}^s (M^l l^{2s} + l^{4s}) \\ \mathbb{E} \left\{ \frac{1}{T} \sum_{j=1}^T \mathbf{F}_{j+\tau}^* [\alpha_j^{(i_1, i_2)}] \mathbf{F}_{j+\tau} - \text{tr} [\alpha_j^{(i_1, i_2)}] \right\}^{2s} &\leq \frac{C^{2s}}{T^s} \sum_{l=1}^s (M^l l^{2s} + l^{4s}). \end{aligned}$$

Substituting the above back to (5.2) and choosing  $s \geq 2$ , we have

$$\mathbf{P}_{1(i_1, i_2)} - \frac{\alpha_j}{2} \delta_{(i_1, i_2)} - \frac{\alpha_{j+\tau}}{2} \delta_{(i_1, i_2)} = o_{a.s.}(1).$$

Again, by the almost sure and uniform convergence of  $\alpha_j$  and  $\alpha_{j+\tau}$  to  $C_0$ , we have

$$\frac{\alpha_j}{2} \delta_{(i_1, i_2)} + \frac{\alpha_{j+\tau}}{2} \delta_{(i_1, i_2)} - C_0 \delta_{(i_1, i_2)} = o_{a.s.}(1).$$

Therefore, we have shown that  $P_1 - \mathbb{E}(P_1) = o_{a.s.}(1)$ . Results for  $i = 2, 3, 4, 5$  can be shown in a similar way.

Denote  $\alpha = \frac{-\frac{cm}{2}}{1 - \frac{c^2 m^2}{2x_1}}$  and  $\beta = \frac{-\frac{cm}{2}}{1 - \frac{c^2 m^2}{4x_1}}$ , then we have  $C_p = \alpha \beta^p$ . Note that  $\mathbf{H}(p\tau) = \mathbf{0}$  for

$p > \lceil q/\tau \rceil$ , and we have, with probability 1 that

$$\begin{aligned}
& \mathbf{W}\mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* \mathbf{W}^* \\
\rightarrow & \left( \frac{1}{2} C_0 + \frac{1}{2} C_2 \right) \mathbf{H}(0) + \left( \frac{3}{2} C_1 + \frac{1}{2} C_3 \right) \mathbf{H}(\tau) + \sum_{p=2}^{\infty} \left( \frac{1}{2} C_{p-2} + C_p + \frac{1}{2} C_{p+2} \right) \mathbf{H}(p\tau) \\
= & \frac{\alpha}{2} \left[ (1 + \beta^2) \mathbf{H}(0) + (3\beta + \beta^3) \mathbf{H}(\tau) + (1 + \beta^2)^2 \sum_{p=2}^{\infty} \beta^{p-2} \mathbf{H}(p\tau) \right] \\
= & \frac{\alpha}{2} \left[ (1 + \beta^2) \mathbf{H}(0) + (3\beta + \beta^3) \mathbf{H}(\tau) + (1 + \beta^2)^2 \sum_{p=2}^{\infty} \beta^{p-2} (\mathbf{H}_L(p\tau) + \mathbf{H}_U(p\tau)) \right] \\
= & \frac{\alpha}{2} \left[ (1 + \beta^2) \mathbf{H}(0) + (3\beta + \beta^3) \mathbf{H}(\tau) + (1 + \beta^2)^2 \sum_{p=2}^{\infty} \beta^{p-2} (\mathbf{J}_L(p\tau) + \mathbf{J}_U(p\tau)) \otimes \mathbf{I}_k \right] \\
= & \frac{\alpha}{2} \left[ (1 + \beta^2) \mathbf{H}(0) + (3\beta + \beta^3) \mathbf{H}(\tau) + (1 + \beta^2)^2 \sum_{p=2}^{\infty} \beta^{p-2} (\mathbf{J}_L^p(\tau) + \mathbf{J}_U^p(\tau)) \otimes \mathbf{I}_k \right] \\
= & \frac{\alpha}{2} \left[ (1 + \beta^2) \mathbf{H}(0) + (3\beta + \beta^3) \mathbf{H}(\tau) + (1 + \beta^2)^2 \left( \mathbf{J}_L^2(\tau) (\mathbf{I} - \beta \mathbf{J}_L(\tau))^{-1} \right. \right. \\
& \left. \left. + \mathbf{J}_U^2(\tau) (\mathbf{I} - \beta \mathbf{J}_U(\tau))^{-1} \right) \otimes \mathbf{I}_k \right].
\end{aligned}$$

Note that

$$\begin{aligned}
\mathbf{J}_L(\tau) (\mathbf{I} - \beta \mathbf{J}_L(\tau))^{-1} &= \frac{1}{\beta} [\mathbf{I} - (\mathbf{I} - \beta \mathbf{J}_L(\tau))] (\mathbf{I} - \beta \mathbf{J}_L(\tau))^{-1} \\
&= \frac{1}{\beta} [(\mathbf{I} - \beta \mathbf{J}_L(\tau))^{-1} - \mathbf{I}] \\
\mathbf{J}_L^2(\tau) (\mathbf{I} - \beta \mathbf{J}_L(\tau))^{-1} &= \frac{1}{\beta} \mathbf{J}_L(\tau) [(\mathbf{I} - \beta \mathbf{J}_L(\tau))^{-1} - \mathbf{I}] \\
&= \frac{1}{\beta^2} [(\mathbf{I} - \beta \mathbf{J}_L(\tau))^{-1} - \mathbf{I}] - \frac{1}{\beta} \mathbf{J}_L(\tau).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{J}_U^2(\tau) (\mathbf{I} - \beta \mathbf{J}_U(\tau))^{-1} &= \frac{1}{\beta} \mathbf{J}_U(\tau) [(\mathbf{I} - \beta \mathbf{J}_U(\tau))^{-1} - \mathbf{I}] \\
&= \frac{1}{\beta^2} [(\mathbf{I} - \beta \mathbf{J}_U(\tau))^{-1} - \mathbf{I}] - \frac{1}{\beta} \mathbf{J}_U(\tau).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& (1 + \beta^2)^2 \left( \mathbf{J}_L^2(\tau)(\mathbf{I} - \beta \mathbf{J}_L(\tau))^{-1} + \mathbf{J}_U^2(\tau)(\mathbf{I} - \beta \mathbf{J}_U(\tau))^{-1} \right) \otimes \mathbf{I}_k \\
= & (1 + \beta^2)^2 \left( \frac{1}{\beta^2} (\mathbf{I} - \beta \mathbf{J}_L(\tau))^{-1} + \frac{1}{\beta^2} (\mathbf{I} - \beta \mathbf{J}_U(\tau))^{-1} \right. \\
& \left. - \frac{2}{\beta^2} \mathbf{I} - \frac{1}{\beta} (\mathbf{J}_L(\tau) + \mathbf{J}_U(\tau)) \right) \otimes \mathbf{I}_k \\
= & \frac{(1 + \beta^2)^2}{\beta^2} \left[ (\mathbf{I} - \beta \mathbf{J}_L(\tau))^{-1} (2\mathbf{I} - \beta \mathbf{J}_L(\tau) - \beta \mathbf{J}_U(\tau)) (\mathbf{I} - \beta \mathbf{J}_U(\tau))^{-1} \right] \otimes \mathbf{I}_k \\
& - \frac{(1 + \beta^2)^2}{\beta^2} \mathbf{H}(0) - \frac{(1 + \beta^2)^2}{\beta} \mathbf{H}(\tau) \\
=: & \frac{(1 + \beta^2)^2}{\beta^2} \mathbf{G}(\tau) - \frac{(1 + \beta^2)^2}{\beta^2} \mathbf{H}(0) - \frac{(1 + \beta^2)^2}{\beta} \mathbf{H}(\tau).
\end{aligned}$$

Hence, we have a.s.

$$\begin{aligned}
& \mathbf{B}_1^* \Lambda \mathbf{W} \mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* \mathbf{W}^* \Lambda^* \mathbf{B}_1 \\
\rightarrow & \frac{\alpha}{2} \mathbf{Q}^{1/2} \left[ \left( 1 + \beta^2 - \frac{(1 + \beta^2)^2}{\beta^2} \right) \mathbf{H}(0) + \left( 3\beta + \beta^3 - \frac{(1 + \beta^2)^2}{\beta} \right) \mathbf{H}(\tau) \right. \\
& \left. + \frac{(1 + \beta^2)^2}{\beta^2} \mathbf{G}(\tau) \right] \mathbf{Q}^{1/2}.
\end{aligned}$$

Last, we want to show that with probability 1,

$$\mathbf{B}_1^* \Lambda \mathbf{W} \mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* \mathbf{M} \mathbf{B}_1 \rightarrow \mathbf{0}$$

and

$$\mathbf{B}_1^* \mathbf{M} \mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* \mathbf{W}^* \Lambda^* \mathbf{B}_1 \rightarrow \mathbf{0}.$$

Note that

$$\mathbf{B}_1^* \Lambda \mathbf{W} \mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* \mathbf{M} \mathbf{B}_1 = \mathbf{B}_1^* \Lambda \mathbf{W} \left( \mathbf{I} - \frac{1}{\ell} \mathbf{B}_2 \mathbf{B}_2^* \mathbf{M} \right)^{-1} \mathbf{B}_1 - \mathbf{B}_1^* \Lambda \mathbf{W} \mathbf{B}_1$$

and that

$$\mathbf{B}_1^* \mathbf{M} \mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* \Lambda^* \mathbf{W} \mathbf{B}_1 = \mathbf{B}_1^* \left( \mathbf{I} - \frac{1}{\ell} \mathbf{M} \mathbf{B}_2 \mathbf{B}_2^* \right)^{-1} \mathbf{W}^* \Lambda^* \mathbf{B}_1 - \mathbf{B}_1^* \mathbf{W}^* \Lambda^* \mathbf{B}_1.$$

Hence, by  $\mathbf{W}\mathbf{B}_1 = o_{a.s.}(\mathbf{1})$  and  $\mathbf{B}_1^*\mathbf{W}^* = o_{a.s.}(\mathbf{1})$ , it suffices to show with probability 1 that,

$$\mathbf{B}_1^*\mathbf{\Lambda}\mathbf{W}\left(\mathbf{I} - \frac{1}{\ell}\mathbf{B}_2\mathbf{B}_2^*\mathbf{M}\right)^{-1}\mathbf{B}_1 \rightarrow \mathbf{0}$$

and

$$\mathbf{B}_1^*\left(\mathbf{I} - \frac{1}{\ell}\mathbf{M}\mathbf{B}_2\mathbf{B}_2^*\right)^{-1}\mathbf{W}^*\mathbf{\Lambda}^*\mathbf{B}_1 \rightarrow \mathbf{0}.$$

By  $\mathbf{B}_1 = \mathbf{\Lambda}(\mathbf{\Lambda}^*\mathbf{\Lambda})^{-1/2}$ , we have

$$\begin{aligned} & \mathbf{B}_1^*\mathbf{\Lambda}\mathbf{W}\left(\mathbf{I} - \frac{1}{\ell}\mathbf{B}_2\mathbf{B}_2^*\mathbf{M}\right)^{-1}\mathbf{B}_1 \\ &= (\mathbf{\Lambda}^*\mathbf{\Lambda})^{1/2}\mathbf{W}\left(\mathbf{I} - \frac{1}{\ell}\mathbf{B}_2\mathbf{B}_2^*\mathbf{M}\right)^{-1}\mathbf{\Lambda}(\mathbf{\Lambda}^*\mathbf{\Lambda})^{-1/2} \\ &= \mathbf{Q}^{1/2}\mathbf{W}\left(\mathbf{I} - \frac{1}{\ell}\mathbf{B}_2\mathbf{B}_2^*\mathbf{M}\right)^{-1}\mathbf{\Lambda}\mathbf{Q}^{-1/2}. \end{aligned}$$

By law of large numbers, we have with probability 1 that,

$$\begin{aligned} & \mathbf{W}\left(\mathbf{I} - \frac{1}{\ell}\mathbf{B}_2\mathbf{B}_2^*\mathbf{M}\right)^{-1}\mathbf{\Lambda} \\ &= \frac{1}{2T} \sum_{j=1}^T (\mathbf{F}_j\mathbf{\varepsilon}_{j+\tau}^* + \mathbf{F}_{j+\tau}\mathbf{\varepsilon}_j^*) \left(\mathbf{I} - \frac{1}{\ell}\mathbf{B}_2\mathbf{B}_2^*\mathbf{M}\right)^{-1}\mathbf{\Lambda} \\ &\rightarrow \mathbf{E}(\mathbf{F}_1\mathbf{\varepsilon}_{1+\tau}^* + \mathbf{F}_{1+\tau}\mathbf{\varepsilon}_1^*) \left(\mathbf{I} - \frac{1}{\ell}\mathbf{B}_2\mathbf{B}_2^*\mathbf{M}\right)^{-1}\mathbf{\Lambda} \\ &= \mathbf{E}\mathbf{E}\left((\mathbf{F}_1\mathbf{\varepsilon}_{1+\tau}^* + \mathbf{F}_{1+\tau}\mathbf{\varepsilon}_1^*) \left(\mathbf{I} - \frac{1}{\ell}\mathbf{B}_2\mathbf{B}_2^*\mathbf{M}\right)^{-1}\mathbf{\Lambda} \middle| \varepsilon_1, \dots, \varepsilon_{\mathbf{T}+\tau}\right) \\ &= o(\mathbf{1}). \end{aligned}$$

Hence, we have with probability 1

$$\mathbf{B}_1^*\mathbf{\Lambda}\mathbf{W}\left(\mathbf{I} - \frac{1}{\ell}\mathbf{B}_2\mathbf{B}_2^*\mathbf{M}\right)^{-1}\mathbf{B}_1 = \mathbf{Q}^{1/2}\mathbf{W}\left(\mathbf{I} - \frac{1}{\ell}\mathbf{B}_2\mathbf{B}_2^*\mathbf{M}\right)^{-1}\mathbf{\Lambda}\mathbf{Q}^{-1/2} = o(\mathbf{1}).$$

Similarly,

$$\mathbf{B}_1^*\left(\mathbf{I} - \frac{1}{\ell}\mathbf{M}\mathbf{B}_2\mathbf{B}_2^*\right)^{-1}\mathbf{W}^*\mathbf{\Lambda}^*\mathbf{B}_1 = o_{a.s.}(\mathbf{1}).$$

Therefore,  $\ell$  should satisfy

$$\begin{aligned} & \det \left| \mathbf{Q}^{1/2} \mathbf{H}(\tau) \mathbf{Q}^{1/2} + \left( \mathbf{B}_1^* (\mathbf{M} - \ell \mathbf{I})^{-1} \mathbf{B}_1 \right)^{-1} + \right. \\ & \frac{\alpha}{2} \mathbf{Q}^{1/2} \left[ \left( 1 + \beta^2 - \frac{(1 + \beta^2)^2}{\beta^2} \right) \mathbf{H}(0) + \left( 3\beta + \beta^3 - \frac{(1 + \beta^2)^2}{\beta} \right) \mathbf{H}(\tau) \right. \\ & \left. \left. + \frac{(1 + \beta^2)^2}{\beta^2} \mathbf{G}(\tau) \right] \mathbf{Q}^{1/2} \right| \rightarrow 0. \end{aligned}$$

Recall  $\mathbf{B}_1 = \mathbf{\Lambda} (\mathbf{\Lambda}^* \mathbf{\Lambda})^{-1/2}$ . Our next goal is to find the limit of

$$\mathbf{B}_1^* (\mathbf{M} - \ell \mathbf{I})^{-1} \mathbf{B}_1.$$

Define  $\mathbf{A} = \mathbf{M} - \ell \mathbf{I}$  and  $\mathbf{A}_k = \mathbf{A} - (\gamma_{k+\tau} + \gamma_{k-\tau}) \gamma_k^* - \gamma_k (\gamma_{k+\tau} + \gamma_{k-\tau})^*$ , then we have the following lemmas, with proofs given in the Appendix.

**Lemma 5.2** *Let  $\mathbf{x} \in \mathbb{C}_1^n := \{\mathbf{x} \in \mathbb{C}^n : \|\mathbf{x}\| = 1\}$  be given. For  $r \geq 1$ , we have*

$$E |\gamma_k^* \mathbf{A}_k^{-1} \mathbf{x}|^{2r} \leq K T^{-r}$$

for some  $K > 0$ .

**Lemma 5.3** *For any  $\mathbf{x}, \mathbf{y} \in \mathbb{C}_1^n$ , we have  $\mathbf{x}^* \mathbf{A}^{-1} \mathbf{y} \rightarrow -\frac{\mathbf{x}^* \mathbf{y}}{\frac{cm}{1-c^2 m^2} + \ell}$  a.s.*

Finally, we have

$$\begin{aligned} & \det \left| \mathbf{Q}^{1/2} \mathbf{H}(\tau) \mathbf{Q}^{1/2} - \left( \frac{cm(\ell)}{1 - c^2 m^2(\ell) + \sqrt{1 - c^2 m^2(\ell)}} + \ell \right) \mathbf{I}_{k(q+1)} \right. \\ & + \frac{\alpha}{2} \mathbf{Q}^{1/2} \left[ \left( 1 + \beta^2 - \frac{(1 + \beta^2)^2}{\beta^2} \right) \mathbf{H}(0) + \left( 3\beta + \beta^3 - \frac{(1 + \beta^2)^2}{\beta} \right) \mathbf{H}(\tau) \right. \\ & \left. \left. + \frac{(1 + \beta^2)^2}{\beta^2} \mathbf{G}(\tau) \right] \mathbf{Q}^{1/2} \right| = 0, \end{aligned}$$

or equivalently

$$\begin{aligned} & \det \left| \frac{\alpha}{2} \left[ \left( 1 + \beta^2 - \frac{(1 + \beta^2)^2}{\beta^2} \right) \mathbf{H}(0) + \left( 3\beta + \beta^3 - \frac{(1 + \beta^2)^2}{\beta} + \frac{2}{\alpha} \right) \mathbf{H}(\tau) \right. \right. \\ & \left. \left. + \frac{(1 + \beta^2)^2}{\beta^2} \mathbf{G}(\tau) \right] - \left( \frac{cm(\ell)}{1 - c^2 m^2(\ell) + \sqrt{1 - c^2 m^2(\ell)}} + \ell \right) \mathbf{Q}^{-1} \right| = 0. \end{aligned} \quad (5.3)$$

When  $\tau > q$ , one has  $\mathbf{H}(\tau) = \mathbf{0}$ ,  $\mathbf{G}(\tau) = 2\mathbf{I}$  and (5.3) reduces to

$$\det \left| \alpha(1 + \beta^2)\mathbf{I} - \left( \frac{cm(\ell)}{1 - c^2m^2(\ell) + \sqrt{1 - c^2m^2(\ell)}} + \ell \right) \mathbf{Q}^{-1} \right| = 0,$$

or

$$\det \left| \frac{cm(\ell)}{1 - c^2m^2(\ell) + \sqrt{1 - c^2m^2(\ell)}} (\mathbf{Q} + \mathbf{I}) + \ell \mathbf{I} \right| = 0.$$

Let  $\lambda$  be an eigenvalue of  $\mathbf{Q}$ , then we have

$$\frac{cm(\ell)}{1 - c^2m^2(\ell) + \sqrt{1 - c^2m^2(\ell)}} \left( 1 + \frac{1}{\lambda} \right) + \frac{\ell}{\lambda} = 0. \quad (5.4)$$

When  $\tau = q$ , (5.3) reduces to

$$\det \left| \alpha(1 + \beta^2)\mathbf{I} + \left( 1 + \frac{\alpha}{2}(3\beta + \beta^3) \right) \mathbf{H}(q) - \left( \frac{cm(\ell)}{1 - c^2m^2(\ell) + \sqrt{1 - c^2m^2(\ell)}} + \ell \right) \mathbf{Q}^{-1} \right| = 0.$$

Writing

$$\mathbf{H}(q) = \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix} \otimes \mathbf{I}_k,$$

one can easily verify that the eigenvalues of  $\mathbf{H}(q)$  are 1,  $-1$  and 0, with multiplicity  $k$ ,  $k$  and  $k(q - 1)$ , respectively.

Suppose that  $\mathbf{H}(q)$  and  $\mathbf{Q}$  are commutative, that is, there is a common orthogonal matrix  $\mathbf{O}$  simultaneously diagonalizing the two matrices, i.e., we have  $\mathbf{H}(q) = \mathbf{O}\mathbf{D}^{\mathbf{H}}\mathbf{O}'$  and  $\mathbf{Q} = \mathbf{O}\mathbf{D}^{\mathbf{Q}}\mathbf{O}'$ , where  $\mathbf{D}^{\mathbf{H}} = \text{diag}[a_1, \dots, a_{k(q+1)}]$  and  $\mathbf{D}^{\mathbf{Q}} = \text{diag}[\lambda_1, \dots, \lambda_{k(q+1)}]$ . Then, (5.3) further reduces to

$$\begin{aligned} & \left( 1 + \frac{\alpha}{2}(3\beta + \beta^3) \right) a_j \\ = & \frac{cm(\ell)}{1 - c^2m^2(\ell) + \sqrt{1 - c^2m^2(\ell)}} \left( 1 + \frac{1}{\lambda_j} \right) + \frac{\ell}{\lambda_j}, \quad j = 1, \dots, k(q + 1). \end{aligned}$$

Substituting  $\alpha = \frac{-\frac{cm}{2}}{1-\frac{c^2m^2}{2x_1}}$  and  $\beta = \frac{-\frac{cm}{2}}{1-\frac{c^2m^2}{4x_1}}$ , for  $j = 1, \dots, k(q+1)$ , we have

$$\begin{aligned} a_j &= \left( \frac{1}{2} + \frac{1}{1 - c^2m^2(\ell) + \sqrt{1 - c^2m^2(\ell)}} \right)^{-1} \times \\ &\quad \left[ \frac{cm(\ell)}{1 - c^2m^2(\ell) + \sqrt{1 - c^2m^2(\ell)}} \left( 1 + \frac{1}{\lambda_j} \right) + \frac{\ell}{\lambda_j} \right] \\ &=: g_j(\ell). \end{aligned} \tag{5.5}$$

Notice that (5.4) is a special case of (5.5) for  $a_j = 0$ .

Note that when  $x$  is outside the support  $[-d(c), d(c)]$  of the LSD of  $\mathbf{M}_n(\tau)$ ,  $m_2(x) \neq 0$ . Hence, we have

$$\begin{aligned} &cm_1(x)((1 - c - cxm_1(x))^2 - c^2x^2m_2^2(x)) \\ &+ x(1 - c^2m_1^2(x) + c^2m_2^2(x))(1 - c - cxm_1(x)) = 0. \end{aligned}$$

Let  $x \downarrow d(c) := d$  and we have  $m_2(x) \rightarrow 0$  and  $m_1(x) \rightarrow m_1(d)$  satisfying

$$\begin{aligned} &cm_1(d)(1 - c - cdm_1(d))^2 + d(1 - c^2m_1^2(d))(1 - c - cdm_1(d)) \\ &= [1 - c - cdm_1(d)][cm_1(d)(1 - c - cdm_1(d)) + d(1 - c^2m_1^2(d))] = 0, \end{aligned}$$

from which we have  $m_1(d) = \frac{1-c-\sqrt{(1-c)^2+8d^2}}{4cd}$ .

Rewrite

$$\begin{aligned} g_j(\ell) &= \frac{cm(\ell)}{\frac{3}{2} - \frac{1}{2}c^2m^2(\ell) + \frac{1}{2}\sqrt{1 - c^2m^2(\ell)}} \left( 1 + \frac{1}{\lambda_j} \right) \\ &\quad + \left( \frac{1}{2} + \frac{1}{1 - c^2m^2(\ell) + \sqrt{1 - c^2m^2(\ell)}} \right)^{-1} \frac{\ell}{\lambda_j} \\ &:= g_{j1}(\ell) + g_{j2}(\ell). \end{aligned}$$

We will show that  $g_j(\ell)$  is increasing over  $(d(c), \infty)$  by showing so are  $g_{j1}$  and  $g_{j2}$ . By definition, over  $(d(c), \infty)$ ,  $m$  is an increasing function taking negative values and  $m^2$  is a decreasing function taking positive values. Hence, it is easy to see that  $g_{j1}(\ell)$  is increasing over  $(d(c), \infty)$ .

For  $g_{j2}(\ell)$ , define  $h(\ell) = 1 - c^2m^2(\ell) + \sqrt{1 - c^2m^2(\ell)}$  and rewrite  $g_{j2}(\ell) = \frac{\ell h(\ell)}{\lambda_j(1 + \frac{h(\ell)}{2})}$ . It is easy

to see that  $h(\ell) > 0$  and  $h'(\ell) > 0$  over  $(d(c), \infty)$ . Hence we have

$$g'_{j2}(\ell) = \frac{[h(\ell) + \ell h'(\ell)](1 + \frac{h(\ell)}{2}) - \ell h(\ell) \frac{h'(\ell)}{2}}{\lambda_j(1 + \frac{h(\ell)}{2})^2} = \frac{h(\ell) + \ell h'(\ell) + \frac{h^2(\ell)}{2}}{\lambda_j(1 + \frac{h(\ell)}{2})^2} > 0.$$

By symmetry,  $g_j(\ell)$  is increasing over  $(-\infty, -d(c))$  as well. Therefore, based on the sign of  $g_j(d(c))$ , we have the following cases to consider.

Case I.  $g_j(d(c)) \geq 0$ :

- i. If  $a_j > g_j(d(c))$ , then (5.5) has one solution in  $(d(c), \infty)$  and no solution in  $(-\infty, -d(c))$ .
- ii. If  $g(d(c)) \geq a_j \geq -g_j(d(c)) = g_j(-d(c))$ , then (5.5) has no solution in  $(d(c), \infty)$  and  $(-\infty, -d(c))$ .
- iii. If  $a_j < -g_j(d(c))$ , then (5.5) has one solution in  $(-\infty, -d(c))$  and no solution in  $(d(c), \infty)$ .

Case II.  $g_j(d(c)) < 0$ :

- i. If  $a_j \geq -g_j(d(c))$ , then (5.5) has one solution in  $(d(c), \infty)$  and no solution in  $(-\infty, -d(c))$ .
- ii. If  $g(d(c)) < a_j < -g_j(d(c)) = g_j(-d(c))$ , then (5.5) has one solution in  $(d(c), \infty)$  and one solution in  $(-\infty, -d(c))$ .
- iii. If  $a_j \leq g_j(d(c))$ , then (5.5) has one solution in  $(-\infty, -d(c))$  and no solution in  $(d(c), \infty)$ .

**Remark 5.1** In real application, compared with the noise component, the loading matrix  $\mathbf{\Lambda}$  dominates. As a result, all the eigenvalues of  $\mathbf{Q} = \mathbf{\Lambda}^* \mathbf{\Lambda}$  are large (more precisely, they are of the same order as  $n$ ). Hence, we can assume that  $\mathbf{Q}^{-1} = \mathbf{0}$ . Thus the commutative assumption of  $\mathbf{H}(q)$  and  $\mathbf{Q}$  can be relaxed. Moreover, under this case, we always have  $g_j(d(c)) < 0$ .

Notice that all the eigenvalues of  $\mathbf{H}(q+1)$  are 0, while for  $\mathbf{H}(q)$ ,  $k$  eigenvalues are 1 and  $k$  eigenvalues are  $-1$ , with the rest being 0. Making use of such difference and applying the above analysis to the cases that  $\tau = q$  and  $\tau = q+1$  gives an estimate of  $q$ . Together with the estimation of  $k(q+1)$ , we easily obtain the estimate of  $k$ . A numerical demonstration is given in the simulation.



## 6 Estimate of $\sigma^2$

The above estimation is based on the assumption that  $\sigma^2$ , the variance of the noise part is given. In practice, it is often the case that  $\sigma^2$  is unknown. To this end, we can first estimate  $\sigma^2$  by employing the properties of the MP law. More precisely, we can estimate the left boundary of the support of the MP law by the smallest sample eigenvalue of  $\Phi_n(0)$ , say  $\hat{\lambda}_{min}$ , and then estimate the right boundary by  $\frac{(1+\sqrt{c})^2}{(1-\sqrt{c})^2} \hat{\lambda}_{min}$ . Finally,  $\sigma^2$  can be estimated by taking the sample mean of the sample eigenvalues of  $\Phi(0)$  that lie within the interval  $[\hat{\lambda}_{min}, \frac{(1+\sqrt{c})^2}{(1-\sqrt{c})^2} \hat{\lambda}_{min}]$ . As shown in the simulation, our estimation of  $k$  and  $q$  still works well with such estimation of  $\sigma^2$ .

## 7 Simulation

Table 1 presents a simulation about the result discussed above, displaying the largest 13 absolute values of the eigenvalues for lags  $\tau$  from 0 to 5. Here

$$\mathbf{R}_t = \sum_{i=0}^q \Lambda_i \mathbf{f}_{t-i} + \mathbf{e}_t, \quad t = 1, \dots, T \quad (7.1)$$

where  $\mathbf{f}_t$ 's are factors of length  $k$ ;  $\Lambda_i, i = 0, \dots, q$  is a constant time-invariant matrix of size  $n \times k$ ,  $\mathbf{e}_t$  is the error term and  $q$  is the lag of the model. In addition, assume that:  $\mathbf{e}_t$  are i.i.d. random variables with  $\mathbf{e}_t \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_n)$  and  $\mathbf{f}_t$  are i.i.d. random variables with  $\mathbf{f}_t \sim N(\mathbf{0}, \sigma_f^2 \mathbf{I}_k)$ , independent of  $\mathbf{e}_t$ .  $\Lambda_i = \begin{bmatrix} \Lambda_1^i & \Lambda_2^i & \dots & \Lambda_k^i \end{bmatrix}$  where  $\Lambda_j^i, i = 0, \dots, q; j = 1, \dots, k$  is a vector of length  $n$  and is given by  $\Lambda_j^i = \beta \mathbf{1}_n + \varepsilon_{ij}$  where  $\mathbf{1}_n$  is a vector of 1's and  $\varepsilon_{ij}$  are i.i.d. random variables with  $\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2 \mathbf{I}_n)$ . For  $n = 450, T = 500, k = 2, q = 2, \beta = 1.0, \sigma_f^2 = 4, \sigma^2 = 1$  and  $\sigma_\varepsilon^2 = 0.25$ , we have  $c = 0.9, b_c = (1 + \sqrt{c})^2 = 3.7974$  and  $d_c = 1.8573$ . Eigenvalues of  $\mathbf{Q}$  are from 95 to 285, making  $\mathbf{Q}^{-1} \sim \mathbf{0}$ .

When  $\tau = 0$ , using the phase transition point  $b_c = (1 + \sqrt{c})^2 = 3.7974$ , we see that the number of spotted spikes is 6, which estimates  $k(q+1)$ . When for  $\tau = q+1$ , we have  $\mathbf{H}(\tau) = \mathbf{0}$ . Moreover, as  $\mathbf{Q}^{-1} \sim \mathbf{0}$ , we have  $g_j(d(c)) \sim -0.4284 < 0$ . That is, our Case II (ii) applies for all the  $k(q+1)$  0 eigenvalues of  $\mathbf{H}(q+1)$ , making the number of spikes  $2k(q+1)$  as verified

by applying the phase transition point  $d_c = 1.8573$ . For  $\tau = q$ ,  $\mathbf{H}(\tau)$  has  $k$  eigenvalues of 1,  $k$  eigenvalues of  $-1$  and  $k(q-1)$  eigenvalues of 0 with Case II (i),(iii) and (ii) applicable, respectively. Thus, we have  $k + k + 2k(q-1) = 2kq < 2k(q+1)$  eigenvalues in this case. Again, this agrees with the use of the phase transition point  $d_c = 1.8573$ . In other words, the number of spikes first jumps to  $2k(q+1)$  at  $\tau = 3$  which estimates  $q+1$ . The estimate of  $k$  is obvious.

When  $\sigma^2 = 1$  is unknown, using technique as in Section 6, one has  $\hat{\sigma}^2 = 0.985$ . It then follows that  $\hat{b}_c = (1 + \sqrt{c})^2 \hat{\sigma}^2 = 3.7404$ ,  $\hat{d}_c = 1.8433$ , which gives the same estimates as above.

## A Some proofs

### A.1 Proof of Lemma 5.1

Define  $\mathbf{M}_k = \mathbf{M} - \gamma_k(\gamma_{k+\tau} + \gamma_{k-\tau})^* - (\gamma_{k+\tau} + \gamma_{k-\tau})\gamma_k^*$ , and

$$\mathbf{M}_{k,k+\tau,\dots,k+l\tau} = \mathbf{M}_{k,k+\tau,\dots,k+(l-1)\tau} - \gamma_{k+(l+1)\tau}\gamma_{k+l\tau}^* - \gamma_{k+l\tau}\gamma_{k+(l+1)\tau}^*, l \geq 1.$$

Suppose that  $i \geq j$ , then we have

$$\begin{aligned} & \gamma_i^* \mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M} \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* \gamma_j \\ = & \gamma_i^* \mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M}_j \mathbf{B}_2 - \mathbf{B}_2^* (\gamma_{j+\tau} + \gamma_{j-\tau}) \gamma_j^* \mathbf{B}_2 - \mathbf{B}_2^* \gamma_j (\gamma_{j+\tau} + \gamma_{j-\tau})^* \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* \gamma_j \\ = & \frac{\gamma_i^* \mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M}_j \mathbf{B}_2 - \mathbf{B}_2^* (\gamma_{j+\tau} + \gamma_{j-\tau}) \gamma_j^* \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* \gamma_j}{1 - (\gamma_{j+\tau} + \gamma_{j-\tau}) \mathbf{B}_2 \left( \ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M}_j \mathbf{B}_2 - \mathbf{B}_2^* (\gamma_{j+\tau} + \gamma_{j-\tau}) \gamma_j^* \mathbf{B}_2 \right)^{-1} \mathbf{B}_2^* \gamma_j} \\ = & \frac{\gamma_i^* \mathbf{B}_2 \left( (\ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M}_j \mathbf{B}_2)^{-1} + \frac{(\ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M}_j \mathbf{B}_2)^{-1} \mathbf{B}_2^* (\gamma_{j+\tau} + \gamma_{j-\tau}) \gamma_j^* \mathbf{B}_2 (\ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M}_j \mathbf{B}_2)^{-1}}{1 - \gamma_j^* \mathbf{B}_2 (\ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M}_j \mathbf{B}_2)^{-1} \mathbf{B}_2^* (\gamma_{j+\tau} + \gamma_{j-\tau})} \right) \mathbf{B}_2^* \gamma_j}{1 - (\gamma_{j+\tau} + \gamma_{j-\tau}) \mathbf{B}_2 \left( (\ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M}_j \mathbf{B}_2)^{-1} + \frac{(\ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M}_j \mathbf{B}_2)^{-1} \mathbf{B}_2^* (\gamma_{j+\tau} + \gamma_{j-\tau}) \gamma_j^* \mathbf{B}_2 (\ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M}_j \mathbf{B}_2)^{-1}}{1 - \gamma_j^* \mathbf{B}_2 (\ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M}_j \mathbf{B}_2)^{-1} \mathbf{B}_2^* (\gamma_{j+\tau} + \gamma_{j-\tau})} \right) \mathbf{B}_2^* \gamma_j} \\ = & \frac{\gamma_i^* \mathbf{B}_2 \left( (\ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M}_j \mathbf{B}_2)^{-1} + (\ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M}_j \mathbf{B}_2)^{-1} \mathbf{B}_2^* (\gamma_{j+\tau} + \gamma_{j-\tau}) \gamma_j^* \mathbf{B}_2 (\ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M}_j \mathbf{B}_2)^{-1} \right) \mathbf{B}_2^* \gamma_j}{1 - (\gamma_{j+\tau} + \gamma_{j-\tau}) \mathbf{B}_2 \left( (\ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M}_j \mathbf{B}_2)^{-1} + (\ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M}_j \mathbf{B}_2)^{-1} \mathbf{B}_2^* (\gamma_{j+\tau} + \gamma_{j-\tau}) \gamma_j^* \mathbf{B}_2 (\ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M}_j \mathbf{B}_2)^{-1} \right) \mathbf{B}_2^* \gamma_j} + o_{a.s.}(1) \end{aligned}$$

$\tau = 0$	$\tau = 1$	$\tau = 2$	$\tau = 3$	$\tau = 4$	$\tau = 5$
<b>9704.4931</b>	<b>8434.1173</b>	<b>3980.6367</b>	<b>609.2927</b>	<b>378.8187</b>	<b>228.8273</b>
<b>527.1865</b>	<b>331.6144</b>	<b>243.0448</b>	<b>73.5304</b>	<b>76.9916</b>	<b>94.6830</b>
<b>427.3869</b>	<b>329.2753</b>	<b>218.7383</b>	<b>42.5568</b>	<b>68.1589</b>	<b>57.6630</b>
<b>403.4414</b>	<b>288.7607</b>	<b>202.1325</b>	<b>33.1072</b>	<b>39.1645</b>	<b>54.3620</b>
<b>376.2584</b>	<b>18.9295</b>	<b>24.7685</b>	<b>22.2658</b>	<b>30.8606</b>	<b>29.3485</b>
<b>326.3144</b>	<b>14.7776</b>	<b>20.1664</b>	<b>20.4441</b>	<b>25.9461</b>	<b>24.6076</b>
3.6318	<b>6.2141</b>	<b>10.2506</b>	<b>12.0770</b>	<b>10.2241</b>	<b>14.8080</b>
3.5730	<b>5.1227</b>	<b>9.8122</b>	<b>9.8203</b>	<b>8.1420</b>	<b>7.2516</b>
3.5444	1.7992	1.8013	<b>7.4750</b>	<b>5.6396</b>	<b>5.4430</b>
3.4737	1.7897	1.7931	<b>4.4313</b>	<b>5.0729</b>	<b>4.2076</b>
3.4352	1.7369	1.7521	<b>4.0153</b>	<b>4.7119</b>	<b>3.6854</b>
3.3890	1.6892	1.7058	<b>3.4722</b>	<b>2.6857</b>	<b>3.2904</b>
3.3834	1.6882	1.7047	1.7732	1.7465	1.7749

Table 1: Absolute values of the largest eigenvalues of the empirical covariance matrix at various lags with parameters:  $n = 450$ ,  $T = 500$ ,  $k = 2$ ,  $q = 2$ ,  $\beta = 1.0$ ,  $\sigma_f^2 = 4$ ,  $\sigma^2 = 1$  and  $\sigma_\varepsilon^2 = 0.25$ . Note that  $c = 0.9$ ,  $b_c = (1 + \sqrt{c})^2 = 3.7974$  and  $d_c = 1.8573$ . When  $\sigma^2 = 1$  is unknown, one has  $\hat{\sigma}^2 = 0.985$ ,  $\hat{b}_c = (1 + \sqrt{c})^2 \hat{\sigma}^2 = 3.7404$  and  $\hat{d}_c = 1.8433$ .

$$= \begin{cases} \frac{-\frac{cm}{2}}{1 - \frac{c^2 m^2}{2x_1}} + o_{a.s.}(1), & i = j \\ \frac{-\frac{cm}{2}}{1 - \frac{c^2 m^2}{2x_1}} \gamma_i^* \mathbf{B}_2 (\ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M}_j \mathbf{B}_2)^{-1} \mathbf{B}_2^* (\gamma_{j+\tau} + \gamma_{j-\tau}) + o_{a.s.}(1), & \text{otherwise.} \end{cases}$$

Next, we have

[illegible]

Note that  $\left| \frac{-\frac{cm}{2}}{1-\frac{c^2m^2}{4x_1}} \right| = \left| \frac{a}{x_1} \right| < 1$ , by induction, we have

$$\gamma_i^* \mathbf{B}_2 (\ell \mathbf{I} - \mathbf{B}_2^* \mathbf{M}_j \mathbf{B}_2)^{-1} \mathbf{B}_2^* \gamma_{j-\tau} = o_{a.s.}(1).$$

Then result then follows by induction. By symmetry, it holds when  $i < j$ . The proof of the lemma is complete.

## A.2 Proof of Lemma 5.2

Let  $\mathbf{A}_k^{-1} \mathbf{x} = \mathbf{b} = (b_1, \dots, b_n)'$ . Noting  $|\varepsilon_{ij}| < C$  and  $\mathbb{E}|\varepsilon_{ij}|^2 = 1$ , we have

$$\begin{aligned} \mathbb{E}(\gamma_k^* \mathbf{A}_k^{-1} \mathbf{x})^{2r} &= \frac{1}{2^r T^r} \mathbb{E} \left( \sum_{i=1}^n \varepsilon_{ki} b_i \right)^{2r} \\ &\leq \frac{1}{2^r T^r} \mathbb{E} \sum_{l=1}^r \sum_{1 \leq j_1 < \dots < j_l \leq n} \sum_{i_1 + \dots + i_l = 2r} \frac{(2r)!}{i_1! \dots i_l!} \varepsilon_{kj_1}^{i_1} b_{j_1}^{i_1} \dots \varepsilon_{kj_l}^{i_l} b_{j_l}^{i_l} \\ &= \frac{1}{2^r T^r} \mathbb{E} \sum_{l=1}^r \sum_{1 \leq j_1 < \dots < j_l \leq n} \sum_{\substack{i_1 + \dots + i_l = 2r, \\ i_1 \geq 2, \dots, i_l \geq 2}} \frac{(2r)!}{i_1! \dots i_l!} \varepsilon_{kj_1}^{i_1} b_{j_1}^{i_1} \dots \varepsilon_{kj_l}^{i_l} b_{j_l}^{i_l} \\ &\leq \frac{K}{2^r T^r} \mathbb{E} \sum_{l=1}^r \sum_{1 \leq j_1 < \dots < j_l \leq n} \sum_{\substack{i_1 + \dots + i_l = 2r, \\ i_1 \geq 2, \dots, i_l \geq 2}} \frac{(2r)!}{i_1! \dots i_l!} |b_{j_1}|^{i_1} \dots |b_{j_l}|^{i_l}. \end{aligned}$$

By  $\sum_{j=1}^n |b_j|^2 = \|\mathbf{A}_k^{-1} \mathbf{x}\|^2$  and Cauchy-Schwartz inequality, we have

$$\begin{aligned} &\sum_{1 \leq j_1 < \dots < j_l \leq n} \sum_{\substack{i_1 + \dots + i_l = 2r, \\ i_1 \geq 2, \dots, i_l \geq 2}} \frac{(2r)!}{i_1! \dots i_l!} |b_{j_1}|^{i_1} \dots |b_{j_l}|^{i_l} \\ &\leq \sum_{\substack{i_1 + \dots + i_l = 2r, \\ i_1 \geq 2, \dots, i_l \geq 2}} \frac{(2r)!}{i_1! \dots i_l!} \left( \sum_{j=1}^n |b_j|^2 \right)^r \\ &\leq l^{2r} \|\mathbf{A}_k^{-1}\|_o^{2r} \|\mathbf{x}\|^{2r} \\ &\leq \frac{l^{2r}}{\eta^{2r}}. \end{aligned}$$

Here  $\eta := \ell - d_c > 0$ . Therefore, we have

$$\mathbb{E}|\gamma_k^* \mathbf{A}_k^{-1} \mathbf{x}|^{2r} \leq K T^{-r}$$

for some  $K > 0$ . The proof of the lemma is complete.

### A.3 Proof of Lemma 5.3

PROOF. Let  $\mathbf{x}, \mathbf{y} \in \mathbb{C}_1^n$  be given. Define  $\mathbf{A}_{k,k+\tau} = \mathbf{A}_k - \gamma_{k+\tau}\gamma_{k+2\tau}^* - \gamma_{k+2\tau}\gamma_{k+\tau}^*$ ,

$\tilde{\mathbf{A}}_k = \mathbf{A}_k + \gamma_k(\gamma_{k+\tau} + \gamma_{k-\tau})^*$  and  $\tilde{\mathbf{A}}_{k,k+\tau} = \mathbf{A}_k - \gamma_{k+2\tau}\gamma_{k+\tau}^*$ . First we have

$$\begin{aligned}
& \mathbf{x}^* \mathbf{A}^{-1} \mathbf{y} - \mathbf{E} \mathbf{x}^* \mathbf{A}^{-1} \mathbf{y} \\
&= \mathbf{x}^* \sum_{k=1}^T (\mathbf{E}_k - \mathbf{E}_{k-1}) (\mathbf{A}^{-1} - \mathbf{A}_k^{-1}(\ell)) \mathbf{y} \\
&= \sum_{k=1}^T (\mathbf{E}_k - \mathbf{E}_{k-1}) \left( -\frac{\mathbf{x}^* \tilde{\mathbf{A}}_k^{-1} (\gamma_{k+\tau} + \gamma_{k-\tau}) \gamma_k^* \tilde{\mathbf{A}}_k^{-1} \mathbf{y}}{1 + \gamma_k^* \tilde{\mathbf{A}}_k^{-1} (\gamma_{k+\tau} + \gamma_{k-\tau})} - \frac{\mathbf{x}^* \mathbf{A}_k^{-1} \gamma_k (\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1} \mathbf{y}}{1 + (\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1} \gamma_k} \right) \\
&\equiv \sum_{k=1}^T (\mathbf{E}_k - \mathbf{E}_{k-1}) (-\alpha_{k1} - \alpha_{k2}).
\end{aligned}$$

Using Lemma 2.1, we have, for  $i = 1, 2$

$$\begin{aligned}
& \mathbf{E} \left| \sum_{k=1}^T (\mathbf{E}_k - \mathbf{E}_{k-1}) \alpha_{ki} \right|^{2l} \\
&\leq K_l \left[ \mathbf{E} \left( \sum_{k=1}^T \mathbf{E}_{k-1} |(\mathbf{E}_k - \mathbf{E}_{k-1}) \alpha_{ki}|^2 \right)^l + \sum_{k=1}^T \mathbf{E} |(\mathbf{E}_k - \mathbf{E}_{k-1}) \alpha_{ki}|^{2l} \right] \\
&\leq K'_l \left[ \mathbf{E} \left( \sum_{k=1}^T \mathbf{E}_{k-1} |\mathbf{E}_k \alpha_{ki}|^2 + \sum_{k=1}^T \mathbf{E}_{k-1} |\alpha_{ki}|^2 \right)^l + \sum_{k=1}^T \mathbf{E} |\mathbf{E}_k \alpha_{ki}|^{2l} + \sum_{k=1}^T \mathbf{E} |\mathbf{E}_{k-1} \alpha_{ki}|^{2l} \right] \\
&\leq 2^l K'_l \left[ \mathbf{E} \left( \sum_{k=1}^T \mathbf{E}_{k-1} |\alpha_{ki}|^2 \right)^l + \sum_{k=1}^T \mathbf{E} |\alpha_{ki}|^{2l} \right]. \tag{A.1}
\end{aligned}$$

Note that

$$\mathbf{A}_k^{-1} = (\tilde{\mathbf{A}}_{k,k+\tau} + \gamma_{k+2\tau}\gamma_{k+\tau}^*)^{-1} = \tilde{\mathbf{A}}_{k,k+\tau}^{-1} - \frac{\tilde{\mathbf{A}}_{k,k+\tau}^{-1} \gamma_{k+2\tau} \gamma_{k+\tau}^* \tilde{\mathbf{A}}_{k,k+\tau}^{-1}}{1 + \gamma_{k+\tau}^* \tilde{\mathbf{A}}_{k,k+\tau}^{-1} \gamma_{k+2\tau}}.$$

Hence, we have

$$\gamma_{k+\tau}^* \mathbf{A}_k^{-1} = \gamma_{k+\tau}^* \tilde{\mathbf{A}}_{k,k+\tau}^{-1} - \frac{\gamma_{k+\tau}^* \tilde{\mathbf{A}}_{k,k+\tau}^{-1} \gamma_{k+2\tau} \gamma_{k+\tau}^* \tilde{\mathbf{A}}_{k,k+\tau}^{-1}}{1 + \gamma_{k+\tau}^* \tilde{\mathbf{A}}_{k,k+\tau}^{-1} \gamma_{k+2\tau}} = \frac{\gamma_{k+\tau}^* \tilde{\mathbf{A}}_{k,k+\tau}^{-1}}{1 + \gamma_{k+\tau}^* \tilde{\mathbf{A}}_{k,k+\tau}^{-1} \gamma_{k+2\tau}}.$$

Next, we have

$$\begin{aligned}
\gamma_{k+\tau}^* \tilde{\mathbf{A}}_{k,k+\tau}^{-1} &= \gamma_{k+\tau}^* \mathbf{A}_{k,k+\tau}^{-1} - \frac{\gamma_{k+\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+2\tau} \gamma_{k+\tau}^* \mathbf{A}_{k,k+\tau}^{-1}}{1 + \gamma_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+\tau}} \\
&= \gamma_{k+\tau}^* \mathbf{A}_{k,k+\tau}^{-1} - \frac{cm}{2} \gamma_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1} + R_{k1},
\end{aligned}$$

where

$$\begin{aligned}
R_{k1} &= \frac{cm}{2} \gamma_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1} - \frac{\gamma_{k+\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+\tau} \gamma_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1}}{1 + \gamma_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+\tau}} \\
&= \left( \frac{\frac{cm}{2} - \gamma_{k+\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+\tau} + \frac{cm}{2} \gamma_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+\tau}}{1 + \gamma_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+\tau}} \right) \gamma_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1}.
\end{aligned}$$

Substitute back, we obtain

$$\begin{aligned}
&\gamma_{k+\tau}^* \mathbf{A}_k^{-1} \mathbf{y} \\
&= \frac{\gamma_{k+\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \mathbf{y} - \frac{\gamma_{k+\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+\tau}}{1 + \gamma_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+\tau}} \gamma_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \mathbf{y}}{1 + \gamma_{k+\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+2\tau} - \frac{cm}{2} \gamma_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+2\tau} + R_{k1} \gamma_{k+2\tau}}, \tag{A.2}
\end{aligned}$$

with

$$R_{k1} \gamma_{k+2\tau} = \left( \frac{\frac{cm}{2} - \gamma_{k+\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+\tau} + \frac{cm}{2} \gamma_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+\tau}}{1 + \gamma_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+\tau}} \right) \gamma_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+2\tau}.$$

Note that

$$\begin{aligned}
&\lim_{n \rightarrow \infty} \left| \frac{\gamma_{k+\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+\tau}}{(1 + \gamma_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+\tau})(1 + \gamma_{k+\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+2\tau} - \frac{cm}{2} \gamma_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+2\tau} + R_{k1} \gamma_{k+2\tau})} \right| \\
&= \left| \frac{\frac{cm}{2}}{1 - \frac{cm}{2} \frac{cm}{2x_1}} \right| = \left| \frac{\frac{cm}{2}}{x_1} \right| < 1
\end{aligned}$$

and

$$\lim_{n \rightarrow \infty} 1 + \gamma_{k+\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+2\tau} - \frac{cm}{2} \gamma_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+2\tau} + R_{k1} \gamma_{k+2\tau} = 1 - \frac{cm}{2} \frac{cm}{2x_1},$$

which is bounded. Using induction, we have  $|\gamma_{k+\tau}^* \mathbf{A}_k^{-1} \mathbf{y}| \leq K |\gamma_{k+\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \mathbf{y}|$ .

Similarly, we have  $|\gamma_{k-\tau}^* \mathbf{A}_k^{-1} \mathbf{y}| \leq K |\gamma_{k-\tau}^* \mathbf{A}_{k,k-\tau}^{-1} \mathbf{y}|$ ,  $|\mathbf{x}^* \mathbf{A}_k^{-1} \gamma_{k+\tau}| \leq K |\mathbf{x}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+\tau}|$  and

$$|\mathbf{x}^* \mathbf{A}_k^{-1} \gamma_{k-\tau}| \leq K |\mathbf{x}^* \mathbf{A}_{k,k-\tau}^{-1} \gamma_{k-\tau}|.$$

Therefore, by noting  $(\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1} \gamma_k = o_{a.s.}(1)$ ,  $|\varepsilon_{it}| < C$ ,  $\mathbf{E}|\varepsilon_{it}|^2 = 1$  and  $\mathbf{x}^* \mathbf{A}_k^{-1} \bar{\mathbf{A}}_k^{-1} \mathbf{x}$

being bounded, we have

$$\begin{aligned}
& \mathbb{E} \left( \sum_{k=1}^T \mathbb{E}_{k-1} |\alpha_{k2}|^2 \right)^l \\
& \leq K \mathbb{E} \left( \sum_{k=1}^T \mathbb{E}_{k-1} |\mathbf{x}^* \mathbf{A}_k^{-1} \gamma_k \gamma_{k+\tau} \mathbf{A}_k^{-1} \mathbf{y}|^2 \right)^l \\
& = K \mathbb{E} \left( \sum_{k=1}^T \frac{1}{2T} \mathbb{E}_{k-1} \mathbf{x}^* \mathbf{A}_k^{-1} \bar{\mathbf{A}}_k^{-1} \mathbf{x} |\gamma_{k+\tau} \mathbf{A}_k^{-1} \mathbf{y}|^2 \right)^l \\
& \leq K \max_k \mathbb{E} |\gamma_{k+\tau} \mathbf{A}_{k,k+\tau}^{-1} \mathbf{y}|^{2l} \\
& \leq \frac{K}{T^l}
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{k=1}^T \mathbb{E} |\alpha_{k2}|^{2l} \\
& = \sum_{k=1}^T \mathbb{E} (|\mathbf{x}^* \mathbf{A}_k^{-1} \gamma_k \gamma_{k+\tau} \mathbf{A}_k^{-1} \mathbf{y}|^2)^l \\
& \leq \frac{K}{T^{l-1}} \max_k \mathbb{E} |\gamma_{k+\tau} \mathbf{A}_{k,k+\tau}^{-1} \mathbf{y}|^{2l} \\
& \leq \frac{K}{T^{2l-1}}
\end{aligned}$$

For  $i = 1$ , by  $\tilde{\mathbf{A}}_k^{-1} = \mathbf{A}_k^{-1} - \frac{\mathbf{A}_k^{-1} \gamma_k (\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1}}{1 + (\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1} \gamma_k}$ , we have

$$\begin{aligned}
& \mathbf{x}^* \tilde{\mathbf{A}}_k^{-1} (\gamma_{k+\tau} + \gamma_{k-\tau}) \gamma_k^* \tilde{\mathbf{A}}_k^{-1} \mathbf{y} \\
& = \mathbf{x}^* \left( \mathbf{A}_k^{-1} - \frac{\mathbf{A}_k^{-1} \gamma_k (\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1}}{1 + (\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1} \gamma_k} \right) (\gamma_{k+\tau} + \gamma_{k-\tau}) \gamma_k^* \left( \mathbf{A}_k^{-1} - \frac{\mathbf{A}_k^{-1} \gamma_k (\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1}}{1 + (\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1} \gamma_k} \right) \mathbf{y} \\
& = \mathbf{x}^* \mathbf{A}_k^{-1} (\gamma_{k+\tau} + \gamma_{k-\tau}) \gamma_k^* \mathbf{A}_k^{-1} \mathbf{y} \\
& \quad - \frac{(\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1} (\gamma_{k+\tau} + \gamma_{k-\tau})}{1 + (\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1} \gamma_k} \mathbf{x}^* \mathbf{A}_k^{-1} \gamma_k \gamma_k^* \mathbf{A}_k^{-1} \mathbf{y} \\
& \quad - \frac{\gamma_k^* \mathbf{A}_k^{-1} \gamma_k}{1 + (\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1} \gamma_k} \mathbf{x}^* \mathbf{A}_k^{-1} (\gamma_{k+\tau} + \gamma_{k-\tau}) (\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1} \mathbf{y} \\
& \quad + \frac{\gamma_k^* \mathbf{A}_k^{-1} \gamma_k (\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1} (\gamma_{k+\tau} + \gamma_{k-\tau})}{(1 + (\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1} \gamma_k)^2} \mathbf{x}^* \mathbf{A}_k^{-1} \gamma_k (\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1} \mathbf{y} \\
& =: \alpha_{k11} - \alpha_{k12} - \alpha_{k13} + \alpha_{k14}
\end{aligned}$$



It is easy to see that work on  $\alpha_{k11}$  and  $\alpha_{k14}$  is the same as that on  $\alpha_{k2}$ .

For  $\alpha_{k12}$ , by Cauchy-Schwartz's inequality, we have

$$\begin{aligned}
& \mathbb{E} \left( \sum_{k=1}^T \mathbb{E}_{k-1} |\alpha_{k12}|^2 \right)^l \\
& \leq K \mathbb{E} \left( \sum_{k=1}^T \mathbb{E}_{k-1} |\mathbf{x}^* \mathbf{A}_k^{-1} \gamma_k|^2 \mathbb{E}_{k-1} |\gamma_k^* \mathbf{A}_k^{-1} \mathbf{y}|^2 \right)^l \\
& = K \mathbb{E} \left( \sum_{k=1}^T \frac{1}{4T^2} \mathbb{E}_{k-1} \mathbf{x}^* \mathbf{A}_k^{-1} \bar{\mathbf{A}}_k^{-1} \mathbf{x} \mathbb{E}_{k-1} \mathbf{y}^* \bar{\mathbf{A}}_k^{-1} \mathbf{A}_k^{-1} \mathbf{y} \right)^l \\
& \leq \frac{K}{T^l}
\end{aligned}$$

and by  $|\varepsilon_{it}| < C$ ,  $\mathbb{E}|\varepsilon_{it}|^2 = 1$  and  $\mathbf{x}^* \mathbf{A}_k^{-1} \bar{\mathbf{A}}_k^{-1} \mathbf{x}$  being bounded, we have

$$\begin{aligned}
& \sum_{k=1}^T \mathbb{E} |\alpha_{k12}|^{2l} \\
& \leq K \sum_{k=1}^T \mathbb{E} (|\mathbf{x}^* \mathbf{A}_k^{-1} \gamma_k|^2 |\gamma_k^* \mathbf{A}_k^{-1} \mathbf{y}|^2)^l \\
& \leq \frac{K}{T^{l-1}} \max_k \mathbb{E} |\gamma_k \mathbf{A}_k^{-1} \mathbf{y}|^{2l} \\
& \leq \frac{K}{T^{2l-1}}.
\end{aligned}$$

By the fact that  $|\gamma_{k\pm\tau}^* \mathbf{A}_k^{-1} \mathbf{y}| \leq K |\gamma_{k\pm\tau}^* \mathbf{A}_{k,k\pm\tau}^{-1} \mathbf{y}|$  and  $|\mathbf{x}^* \mathbf{A}_k^{-1} \gamma_{k\pm\tau}| \leq K |\mathbf{x}^* \mathbf{A}_{k,k\pm\tau}^{-1} \gamma_{k\pm\tau}|$ , the similar result for  $\alpha_{k13}$  follows by the same reason.

Substituting all the above results into (A.1) and choosing  $l$  large enough, we have  $\mathbf{x}^* \mathbf{A}^{-1} \mathbf{y} - \mathbb{E} \mathbf{x}^* \mathbf{A}^{-1} \mathbf{y} \rightarrow 0$  a.s.

Next, we want to show the convergence of  $\mathbb{E} \mathbf{x}^* \mathbf{A}^{-1} \mathbf{y}$ .

By

$$\mathbf{A} = \sum_{k=1}^T (\gamma_k \gamma_{k+\tau}^* + \gamma_{k+\tau} \gamma_k^*) - \ell \mathbf{I}_n$$

we have

$$\mathbf{I}_n = \sum_{k=1}^T (\gamma_k \gamma_{k+\tau}^* \mathbf{A}^{-1} + \gamma_{k+\tau} \gamma_k^* \mathbf{A}^{-1}) - \ell \mathbf{A}^{-1}.$$

Multiplying  $\mathbf{x}^*$  from left and  $\mathbf{y}$  from right and taking expectation, we obtain

$$\begin{aligned}\mathbf{x}^* \mathbf{y} &= \sum_{k=1}^T (\mathbf{E} \mathbf{x}^* \gamma_k \gamma_{k+\tau}^* \mathbf{A}^{-1} \mathbf{y} + \mathbf{E} \mathbf{x}^* \gamma_{k+\tau} \gamma_k^* \mathbf{A}^{-1} \mathbf{y}) - \ell \mathbf{E} \mathbf{x}^* \mathbf{A}^{-1} \mathbf{y} \\ &= \sum_{k=1}^T \mathbf{E} \mathbf{x}^* (\gamma_{k+\tau} + \gamma_{k-\tau}) \gamma_k^* \mathbf{A}^{-1} \mathbf{y} - \ell \mathbf{E} \mathbf{x}^* \mathbf{A}^{-1} \mathbf{y}.\end{aligned}\tag{A.3}$$

By  $\mathbf{A} = \tilde{\mathbf{A}}_k + (\gamma_{k+\tau} + \gamma_{k-\tau}) \gamma_k^*$ ,  $\tilde{\mathbf{A}}_k = \mathbf{A}_k + \gamma_k (\gamma_{k+\tau} + \gamma_{k-\tau})^*$ ,  $(\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1} \gamma_k = o_{a.s.}(1)$ ,  $\gamma_k^* \mathbf{A}_k^{-1} \gamma_k = \frac{cm}{2} + o_{a.s.}(1)$ , we have

$$\begin{aligned}& \gamma_k^* \mathbf{A}^{-1} \mathbf{y} \\ &= \frac{\gamma_k^* \tilde{\mathbf{A}}_k^{-1} \mathbf{y}}{1 + \gamma_k^* \tilde{\mathbf{A}}_k^{-1} (\gamma_{k+\tau} + \gamma_{k-\tau})} \\ &= \frac{\gamma_k^* \left( \mathbf{A}_k^{-1} - \frac{\mathbf{A}_k^{-1} \gamma_k (\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1}}{1 + (\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1} \gamma_k} \right) \mathbf{y}}{1 + \gamma_k^* \left( \mathbf{A}_k^{-1} - \frac{\mathbf{A}_k^{-1} \gamma_k (\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1}}{1 + (\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1} \gamma_k} \right) (\gamma_{k+\tau} + \gamma_{k-\tau})} \\ &= \frac{\gamma_k^* \mathbf{A}_k^{-1} \mathbf{y} + \gamma_k^* \mathbf{A}_k^{-1} \mathbf{y} (\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1} \gamma_k - \gamma_k^* \mathbf{A}_k^{-1} \gamma_k (\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1} \mathbf{y}}{[1 + (\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1} \gamma_k][1 + \gamma_k^* \mathbf{A}_k^{-1} (\gamma_{k+\tau} + \gamma_{k-\tau})] - \gamma_k^* \mathbf{A}_k^{-1} \gamma_k (\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1} (\gamma_{k+\tau} + \gamma_{k-\tau})} \\ &= \frac{-\frac{cm}{2}}{1 - \frac{c^2 m^2}{2x_1}} (\gamma_{k+\tau} + \gamma_{k-\tau})^* \mathbf{A}_k^{-1} \mathbf{y} + o_{a.s.}(1).\end{aligned}\tag{A.4}$$

Next, we have

$$\begin{aligned}& \gamma_{k+\tau}^* \mathbf{A}_k^{-1} \mathbf{y} \mathbf{x}^* \gamma_{k+\tau} \\ &= \left( \frac{(1 + \gamma_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+\tau}) \gamma_{k+\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \mathbf{y} - \gamma_{k+\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+\tau} \gamma_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \mathbf{y}}{(1 + \gamma_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+\tau})(1 + \gamma_{k+\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+2\tau}) - \gamma_{k+\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+\tau} \gamma_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+2\tau}} \right) \mathbf{x}^* \gamma_{k+\tau} \\ &= \frac{\gamma_{k+\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \mathbf{y} \mathbf{x}^* \gamma_{k+\tau} + o_{a.s.}(1)}{1 - \gamma_{k+\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+\tau} \gamma_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \gamma_{k+2\tau}} \\ &= \frac{\mathbf{x}^* \mathbf{A}_{k,k+\tau}^{-1} \mathbf{y} + o_{a.s.}(1)}{2T(1 - \frac{c^2 m^2}{4x_1})}.\end{aligned}$$

Similarly, we can show that  $\gamma_{k-\tau}^* \mathbf{A}_k^{-1} \mathbf{y} \mathbf{x}^* \gamma_{k-\tau} = \frac{\mathbf{x}^* \mathbf{A}_{k,k-\tau}^{-1} \mathbf{y} + o_{a.s.}(1)}{2T(1 - \frac{c^2 m^2}{4x_1})}$ ,  $\gamma_{k+\tau}^* \mathbf{A}_k^{-1} \mathbf{y} \mathbf{x}^* \gamma_{k-\tau} = o_{a.s.}(1)$

and  $\gamma_{k-\tau}^* \mathbf{A}_k^{-1} \mathbf{y} \mathbf{x}^* \gamma_{k+\tau} = o_{a.s.}(1)$ . Next, we will show that  $\mathbf{E} \mathbf{x}^* \mathbf{A}_{k,k\pm\tau}^{-1} \mathbf{y} - \mathbf{E} \mathbf{x}^* \mathbf{A}^{-1} \mathbf{y} = o(1)$ . By writing

$$\mathbf{E} \mathbf{x}^* \mathbf{A}_{k,k\pm\tau}^{-1} \mathbf{y} - \mathbf{E} \mathbf{x}^* \mathbf{A}^{-1} \mathbf{y} = \mathbf{E} \mathbf{x}^* \mathbf{A}_{k,k\pm\tau}^{-1} \mathbf{y} - \mathbf{E} \mathbf{x}^* \mathbf{A}_k^{-1} \mathbf{y} + \mathbf{E} \mathbf{x}^* \mathbf{A}_k^{-1} \mathbf{y} - \mathbf{E} \mathbf{x}^* \mathbf{A}^{-1} \mathbf{y},$$

it is sufficient to show  $\mathbf{E}\mathbf{x}^*\mathbf{A}_k^{-1}\mathbf{y} - \mathbf{E}\mathbf{x}^*\mathbf{A}^{-1}\mathbf{y} = o(1)$ . Note that

$$\begin{aligned} & \mathbf{E}\mathbf{x}^*\mathbf{A}_k^{-1}\mathbf{y} - \mathbf{E}\mathbf{x}^*\mathbf{A}^{-1}\mathbf{y} \\ = & \mathbf{E}\left(\frac{\mathbf{x}^*\tilde{\mathbf{A}}_k^{-1}(\gamma_{k+\tau} + \gamma_{k-\tau})\gamma_k^*\tilde{\mathbf{A}}_k^{-1}\mathbf{y}}{1 + \gamma_k^*\tilde{\mathbf{A}}_k^{-1}(\gamma_{k+\tau} + \gamma_{k-\tau})}\right) + \mathbf{E}\left(\frac{\mathbf{x}^*\mathbf{A}_k^{-1}\gamma_k(\gamma_{k+\tau} + \gamma_{k-\tau})^*\mathbf{A}_k^{-1}\mathbf{y}}{1 + (\gamma_{k+\tau} + \gamma_{k-\tau})^*\mathbf{A}_k^{-1}\gamma_k}\right) \\ = & \mathbf{E}\alpha_{k1} + \mathbf{E}\alpha_{k2} \end{aligned}$$

Previous calculation shows that  $\mathbf{E}|\alpha_{k1}| = o(1)$  and  $\mathbf{E}|\alpha_{k2}| = o(1)$ . Substituting these back to (A.3) and (A.4), we finish proving the lemma.

## References

- [1] Bai, Z.D., Liu H.X. and Wong, W.K. (2011) Asymptotic properties of eigenmatrices of a large sample covariance matrix. *Ann. Appl. Probab.* 21, 1994–2015.
- [2] Bai, Z.D. and Silverstein, J.W. (1998) No eigenvalues outside the support of the limiting spectral distribution of large-dimensional sample covariance matrices. *Ann. Probab.* 26, 316–345.
- [3] Bai, Z.D. and Silverstein, J.W. (2010) *Spectral Analysis of Large Dimensional Random Matrices*, 2nd ed. Springer Verlag, New York.
- [4] Bai, Z.D. and Yao, J.F. (2008) Central limit theorems for eigenvalues in a spiked population model. *Ann. Inst. H. Poincaré Probab. Statist.* Volume 44, Number 3, 447–474.
- [5] Bai, Z.D. and Wang, C. (2015) A note on the limiting spectral distribution of a symmetrized auto-cross covariance matrix. *Statistics and Probability Letters*, 96, 333 – 340.
- [6] Baik, Jinho and Silverstein, J. W. (2006) Eigenvalues of large sample covariance matrices of spiked population models. *J. Multivariate Anal.* 97(6), 1382-1408.
- [7] Burkholder, D.L. (1973) Distribution function inequalities for martingales. *Ann. Probab.* 1, 19–42.

- [8] Chung, K.L. (2001) *A Course in Probability Theory*, 3rd ed. Academic Press, New York.
- [9] Johnstone, I. (2001) On the distribution of the largest eigenvalue in principal components analysis. *Ann. Statist.* 29, 295 – 327.
- [10] Jin, B. S., Wang, C., Bai, Z.D., Nair, K.K. and Harding, M.C. (2014) Limiting spectral distribution of a symmetrized auto-cross covariance matrix. *Ann. Appl. Probab.* 24, 1199–1225.
- [11] Marčenko, V.A. and Pastur, L.A. (1967) Distribution of eigenvalues for some sets of matrices. *Mat. Sb.* 72, 507 – 536.
- [12] Wang, C., Jin, B. S., Bai, Z.D., Nair, K.K. and Harding, M.C. (2015) Strong limit of the extreme eigenvalues of a symmetrized auto-cross covariance matrix. *Ann. Appl. Probab.* 25, 3624 – 3683.